Frequency Response

RC & RLC Circuits
Measuring Responses

- Define input and output
- Time response and frequency response
Time Response and Frequency Response

- Time response graph showing a complex wave pattern.
- Frequency response graph showing a smooth decline from a peak.

Axes:
- Time (sec) for the left graph.
- Frequency (Hz or rad/sec) for the right graph.
Spectrum Analyzer

RIGOL DSA710 Spectrum Analyzer 100kHz-1GHz

Center Frequency: 315.000000 MHz

Peak1: 315.012120 MHz, -26.50 dBm
Peak2: 314.978800 MHz, -26.87 dBm
Freq Deviation: 16.160 kHz
Carrier Offset: -4.940 kHz
Frequency Response

- Sinusoidal Input, Steady State

\[ e(t) = A \cos(\omega t) \]
\[ v(t) = A \left[ \begin{array}{c} \end{array} \right] \cos(\omega t + \left[ \begin{array}{c} \end{array} \right]) \]
\[ V(j\omega) = \frac{1}{j\omega C R + \frac{1}{j\omega C}} \]

\[ E(j\omega) = \frac{1}{j\omega RC + 1} \]

\[ G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{1}{j\omega RC + 1} \]
Frequency Response

- Sinusoidal Input, Steady State

\[ e(t) = A \cos(\omega t) \]

\[ v(t) = A |G(j\omega)| \cos(\omega t + \angle G(j\omega)) \]
Low Pass RC Circuit

\[ G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{1}{j\omega RC + 1} \]

\[ |G(j\omega)| = \left| \frac{1}{j\omega RC + 1} \right| = \frac{1}{\sqrt{(\omega RC)^2 + 1}} \]

\[ \angle G(j\omega) = \angle \left( \frac{1}{j\omega RC + 1} \right) = -\tan^{-1}(\omega RC) \]
\[ R = 100 \Omega, \quad C = 0.001 F \]

**Magnitude**

- \( \omega = 2\pi \), \( |G(j\omega)| = 0.847 \)
- \( \omega = 6\pi \), \( |G(j\omega)| = 0.469 \)

**Phase (degree)**

- \( \omega = 2\pi \), \( \angle G(j\omega) = -32^\circ \)
- \( \omega = 6\pi \), \( \angle G(j\omega) = -62^\circ \)
1Hz
- dB (deci Bell): $20 \log \text{(magnitude)}$
\[
\left| \frac{1}{j\omega \tau + 1} \right|_\text{dB} = -20 \log_{10} |j\omega \tau + 1| \approx -20 \log_{10} 1 = 0 \quad \omega \ll 1/\tau
\]

\[
\left| \frac{1}{j\omega \tau + 1} \right|_\text{dB} = -20 \log_{10} |j\omega \tau + 1| \approx -20 \log_{10} |\omega \tau| \quad \omega \gg 1/\tau
\]
$1/(s\tau + 1)$
\[
\frac{1}{(s\tau + 1)}
\]

\[
\angle \left( \frac{1}{j\omega\tau + 1} \right) \approx -\angle 1 = 0^\circ \quad \omega \ll 1/\tau
\]

\[
\angle \left( \frac{1}{j\omega\tau + 1} \right) = -\angle (j + 1) = -45^\circ \quad \omega = 1/\tau
\]

\[
\angle \left( \frac{1}{j\omega\tau + 1} \right) \approx -\angle (j\omega\tau) = -90^\circ \quad \omega \gg 1/\tau
\]
\[
\frac{1}{s\tau + 1}
\]
Band Width, Cut Off Frequency

\[ G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{1}{j\omega RC + 1} \]

\[ |G(j\omega)| = \left| \frac{1}{j\omega RC + 1} \right| = \frac{1}{\sqrt{(\omega RC)^2 + 1}} \]

\[ \omega RC = 1 \Rightarrow |G(j\omega)| = \left| \frac{1}{j + 1} \right| = \frac{1}{\sqrt{2}} \Rightarrow 20\log_{10} \frac{1}{\sqrt{2}} = -3dB \]

\[ \omega = \frac{1}{RC} \]
High Pass RC Circuit

\[ V(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \]  
\[ E(j\omega) = \frac{j\omega RC}{j\omega RC + 1} \]  
\[ G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{j\omega RC}{j\omega RC + 1} \]
\[
G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{j\omega RC}{j\omega RC + 1}
\]

\[
|G(j\omega)| = \left| \frac{j\omega RC}{j\omega RC + 1} \right| = \frac{\omega RC}{\sqrt{(\omega RC)^2 + 1}}
\]

\[
\angle G(j\omega) = \angle \left( \frac{1}{j\omega RC + 1} \right) = 90^\circ - \tan^{-1}(\omega RC)
\]
Bode Plot

Magnitude (dB)

Phase (degree)
\[ LC \frac{d^2 v(t)}{dt^2} + RC \frac{dv(t)}{dt} + v(t) = e(t) \]

\[ s^2 + \frac{R}{L} s + \frac{1}{LC} = 0, \ s^2 + 2\alpha s + \omega_0^2 = 0 \]

\[ \alpha = \frac{R}{2L}, \ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0, \]

damping ratio: \( \zeta = \frac{\alpha}{\omega_0}, \ \alpha = \zeta\omega_0 \)
\[ s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\zeta \omega_0 \pm \omega_0 \sqrt{\zeta^2 - 1} \]

\[ \zeta < 1: \quad s = -\zeta \omega_0 \pm j\omega_0 \sqrt{1 - \zeta^2} \]

\[ v(t) = -\frac{1}{\sqrt{1 - \left(\frac{\alpha}{\omega_0}\right)^2}} e^{-\alpha t} \cos(\omega_d t - \phi) + 1, \quad t \geq 0 \]
Example 13.3-4  Network Function with Complex Poles

The network function of a second-order low-pass filter has the form

\[
H(\omega) = \frac{k \omega_0^2}{(j\omega)^2 + j2\zeta\omega_0\omega + \omega_0^2}
\]

This network function depends on three parameters: the dc gain \(k\); the corner frequency \(\omega_0\); and the damping ratio \(\zeta\). For convenience, we consider the case where \(k = 1\). Then, using \(j^2 = -1\), we can write the network function as

\[
H(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j2\zeta\omega_0\omega}
\]

Determine the asymptotic magnitude Bode plot of the second-order low-pass filter when the dc gain is 1.

Solution

The denominator of \(H(\omega)\) contains a new factor, one that involves \(\omega^2\). The asymptotic Bode plot is based on the approximation

\[
(\omega_0^2 - \omega^2) + j2\zeta\omega_0\omega \approx \begin{cases} 
\omega_0^2 & \omega < \omega_0 \\
-\omega^2 & \omega > \omega_0
\end{cases}
\]

Using this approximation, we can express \(H(\omega)\) as

\[
H(\omega) \approx \begin{cases} 
1 & \omega < \omega_0 \\
-\frac{\omega_0^2}{\omega^2} & \omega > \omega_0
\end{cases}
\]

The logarithmic gain is

\[
20 \log_{10}|H(\omega)| \approx \begin{cases} 
0 & \omega < \omega_0 \\
40 \log_{10} \omega_0 - 40 \log_{10} \omega & \omega > \omega_0
\end{cases}
\]
The asymptotic magnitude Bode plot is shown in Figure 13.3-10. The actual magnitude Bode plot and the actual phase Bode plot are shown in Figure 13.3-11. The asymptotic Bode plot is a good approximation to the actual Bode plot when \( \omega \ll \omega_0 \) or \( \omega \gg \omega_0 \). Near \( \omega = \omega_0 \), the asymptotic Bode plot deviates from the actual Bode plot. At \( \omega = \omega_0 \), the value of the asymptotic Bode plot is 0 dB, whereas the value of the actual Bode plot is

\[
H(\omega_0) = \frac{1}{2\zeta}
\]

As this equation and Figure 13.3-11 both show, the deviation between the actual and asymptotic Bode plot near \( \omega = \omega_0 \) depends on \( \zeta \). The frequency \( \omega_0 \) is called the corner frequency. The slope of the asymptotic Bode plot decreases by 40 dB/decade as the frequency increases past \( \omega = \omega_0 \). In terms of the asymptotic Bode plot, the denominator of this network function acts like two poles at \( p = \omega_0 \). If this factor were to appear in the numerator of a network function, it would act like two zeros at \( z = \omega_0 \). The slope of the asymptotic Bode plot would increase by 40 dB/decade as the frequency increased past \( \omega = \omega_0 \).

![Figure 13.3-10](image)

**FIGURE 13.3-10** The asymptotic magnitude Bode plot of the second-order low-pass filter when the dc gain is 1.
\[
1 / \left[ \left( \frac{s}{\omega_0} \right)^2 + \left( 2\zeta \frac{s}{\omega_0} \right) + 1 \right]
\]

\[
\left| \frac{1}{(j\omega / \omega_0)^2 + (2\zeta j\omega / \omega_0) + 1} \right|_{dB}
\approx -20 \log_{10} 1 = 0 \quad \omega \ll \omega_0
\]

\[
\left| \frac{1}{(j\omega / \omega_0)^2 + (2\zeta j\omega / \omega_0) + 1} \right|_{dB}
\approx -20 \log_{10} \left| \frac{\omega}{\omega_0} \right|^2 \quad \omega \gg \omega_0
\]
Low Pass RLC Circuit

\[ V(j\omega) = \frac{1}{j\omega C} \frac{j\omega C}{j\omega L + R + \frac{1}{j\omega C}} \]
\[ E(j\omega) = \frac{1}{(j\omega)^2 LC + (j\omega) RC + 1} \]
Low Pass RLC Circuit

\[ V(j\omega) = \frac{1}{j\omega C} \left( \frac{1}{j\omega L + R + \frac{1}{j\omega C}} \right) \]

\[ E(j\omega) = \frac{1}{(j\omega)^2 LC + (j\omega)RC + 1} \]

\[ G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{1}{(j\omega)^2 LC + (j\omega)RC + 1} = \frac{1/(LC)}{(j\omega)^2 + (R/L)(j\omega) + 1/(LC)} \]

\[ G(j\omega) = \frac{\omega_0^2}{(j\omega)^2 + 2\alpha(j\omega) + \omega_0^2} \]

\[ \alpha = \frac{R}{2L}, \omega_0 = \frac{1}{\sqrt{LC}}, \zeta = \frac{\alpha}{\omega_0} \]
\[ R = 100 \Omega, \; L = 100 \text{mH}, \; C = 0.1 \mu F \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-1} \cdot 10^{-7}}} = 10^4 \]

\[ \alpha = \frac{R}{2L} = \frac{100}{2 \times 10^{-1}} = 500, \; \zeta = \frac{500}{10000} = 0.05 \]

\[ |G(j\omega_0)| = \frac{1}{2\zeta} = 10 = 20 \text{dB} \]