Chapter 2
Second-Order Systems
\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) = f_1(x) \\
\dot{x}_2 &= f_2(x_1, x_2) = f_2(x)
\end{align*}
\]

Let \( x(t) = (x_1(t), x_2(t)) \) be a solution that starts at initial state \( x_0 = (x_{10}, x_{20}) \). The locus in the \( x_1-x_2 \) plane of the solution \( x(t) \) for all \( t \geq 0 \) is a curve that passes through the point \( x_0 \). This curve is called a trajectory or orbit. The \( x_1-x_2 \) plane is called the state plane or phase plane. The family of all trajectories is called the phase portrait. The vector field \( f(x) = (f_1(x), f_2(x)) \) is tangent to the trajectory at point \( x \) because

\[
\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}
\]
Phase Portrait

- Graphical study of second order systems
  \[
  \begin{align*}
  \dot{x}_1 &= f_1(x_1, x_2) \\
  \dot{x}_2 &= f_2(x_1, x_2)
  \end{align*}
  \]

- Phase plane trajectory
- Phase portrait: a family of phase plane trajectories
Phase portrait of a mass-spring system

\[ \ddot{x} + x = 0 \]

\[ x(t) = x_0 \cos t \]

\[ \dot{x}(t) = -x_0 \sin t \]

\[ x^2 + \dot{x}^2 = x_0^2 \]

Figure 2.1: A mass-spring system and its phase portrait
A major class of second-order system

\[ \ddot{x} + f(x, \dot{x}) = 0 \]

with \( x_1 = x \) and \( x_2 = \dot{x} \).

\[ \dot{x}_1 = \dot{x}_2 \]

\[ \dot{x}_2 = -f(x_1, x_2) \]
Singular Points

• Singular point: equilibrium point in the phase plane
• Equilibrium point: a point where the system states can stay forever
  solving $\dot{x} = 0$ for the state $x$, i.e., from solving
  
  $f_1(x_1, x_2) = 0$, $f_2(x_1, x_2) = 0$

• Linear system: usually only one singular point
• Nonlinear system: more than one
A nonlinear second-order system

\[ \ddot{x} + 0.5 \dot{x} + 2x + x^2 = 0 \]
\[ \dot{x} = 0, \ddot{x} = 0 \rightarrow 2x + x^2 = 0 \]
\[ \rightarrow x = 0, -2 \]
Singular Point

• The slope of the phase trajectories

\[
\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}
\]

With the functions \(f_1\) and \(f_2\) assumed to be single valued, there is usually a definite value for this slope at any given point in phase plane. This implies that the phase trajectories will not intersect. At singular points, however, the value of the slope is 0/0, \(i.e.,\) the slope is indeterminate. Many trajectories may intersect at such points, as seen from Figure 2.2. This indeterminacy of the slope accounts for the adjective "singular".
Constructing Phase Portraits

- Analytical Method

\[ g(x_1, x_2, c) = 0 \]

The first technique involves solving equations (2.1) for \( x_1 \) and \( x_2 \) as functions of time \( t \), \textit{i.e.},

\[ x_1(t) = g_1(t) \quad x_2(t) = g_2(t) \]

and then eliminating time \( t \) from these equations, leading to a functional relation in the form of (2.6). This technique was already illustrated in Example 2.1.
Constructing Phase Portraits

• **Analytical Method**

The second technique, on the other hand, involves directly eliminating the time variable, by noting that

\[
\frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}
\]

and then solving this equation for a functional relation between \(x_1\) and \(x_2\). Let us use
Mass-spring system

\[ \ddot{x} + x = 0 \]

\[ x_1 = x, \quad x_2 = \dot{x} \]

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = \ddot{x} = -x = -x_1 \]

\[ \frac{dx_2}{dx_1} = \frac{-x_1}{x_2} \rightarrow x_2 \frac{dx_2}{dx_1} + x_1 = 0 \]

\[ \rightarrow \int x_2 \frac{dx_2}{dx_1} dx_1 + \int x_1 dx_1 = C \rightarrow x_2^2 + x_1^2 = x_0^2 \]
A satellite control system

\[
\ddot{\theta} = u
\]

\[
u(t) = \begin{cases} 
-U & \text{if } \theta > 0 \\
U & \text{if } \theta < 0 
\end{cases}
\]
A satellite control system

\[ \theta < 0 \]

\[ \ddot{\theta} = U \rightarrow \ddot{\theta} d\theta = \frac{d\dot{\theta}}{dt} d\theta = Ud\theta \rightarrow \dot{\theta}d\dot{\theta} = Ud\theta \]

\[ \dot{\theta}^2 = U\theta + c_1 \]
A satellite control system

$$\theta > 0$$

$$\frac{1}{2} \dot{\theta}^2 = -U \theta + c_1$$
The Method of Isoclines

locus of the points with a given tangent slope. defined to be

\[
\frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \alpha
\]

This is to say that points on the curve

\[
f_2(x_1, x_2) = \alpha f_1(x_1, x_2)
\]

all have the same tangent slope \( \alpha \).
The Method of Isoclines

- Mass-spring system

\[
\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = \alpha \rightarrow x_1 + \alpha x_2 = 0
\]
The Van der Pol equation

\[ \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \]

\[ \frac{d\dot{x}}{dt} dx + \mu(x^2 - 1)\dot{x} dx + x dx = 0 \]

\[ \dot{x} \frac{d\dot{x}}{dx} + \mu(x^2 - 1)\dot{x} + x = 0 \rightarrow \frac{d\dot{x}}{dx} = \frac{-\mu(x^2 - 1)\dot{x} - x}{\dot{x}} = \alpha \]

\[ \dot{x} = \frac{x}{(\mu - \mu x^2) - \alpha} \]
The Van der Pol equation
Phase Plane Analysis of Linear Systems

• Linear 2\textsuperscript{nd} order system

\[
\dot{x}_1 = ax_1 + bx_2 \\
\dot{x}_2 = cx_1 + dx_2
\]

\[\dot{x} = Ax\]
Linear 2\textsuperscript{nd} Order System

\begin{align*}
\dot{x} &= Ax \rightarrow x(t) = e^{At} x(0) \\
\dot{z} &= J_r z, \; J_r : \text{Jordan form of } A \\
A &= MJ_r M^{-1} \\
z(t) &= e^{J_r t} z(0)
\end{align*}
Linear 2\textsuperscript{nd} Order System

\[ J_r : \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda \\ k \end{bmatrix}, \]

\[ \begin{bmatrix} \alpha + j \beta & 0 \\ 0 & \alpha - j \beta \end{bmatrix} \rightarrow \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \]
Case 1. Both eigenvalues are real: $\lambda_1 \neq \lambda_2 \neq 0$

$$M = [v_1, v_2]$$

$v_1$ & $v_2$ are the real eigenvectors associated with $\lambda_1$ & $\lambda_2$

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2$$

$$z_1(t) = z_{10} e^{\lambda_1 t}, \quad z_2(t) = z_{20} e^{\lambda_2 t}$$

$$z_2 = cz_1^{\lambda_2/\lambda_1}, \quad c = z_{20}/(z_{10})^{\lambda_2/\lambda_1}$$

The shape of the phase portrait depends on the signs of $\lambda_1$ and $\lambda_2$
\[ \lambda_2 < \lambda_1 < 0 \]

\[ e^{\lambda_1 t} \text{ and } e^{\lambda_2 t} \text{ tend to zero as } t \to \infty \]

\[ e^{\lambda_2 t} \text{ tends to zero faster than } e^{\lambda_1 t} \]

Call \( \lambda_2 \) the fast eigenvalue (\( v_2 \) the fast eigenvector) and \( \lambda_1 \) the slow eigenvalue (\( v_1 \) the slow eigenvector)

The trajectory tends to the origin along the curve

\[ z_2 = cz_1^{\lambda_2/\lambda_1} \text{ with } \lambda_2/\lambda_1 > 1 \]

\[ \frac{dz_2}{dz_1} = c \frac{\lambda_2}{\lambda_1} z_1^{[(\lambda_2/\lambda_1)-1]} \]
Stable Node

\[ \lambda_2 > \lambda_1 > 0 \]

Reverse arrowheads

Reverse arrowheads \[\implies\] Unstable Node
\[ \lambda_2 < 0 < \lambda_1 \]

\[ e^{\lambda_1 t} \to \infty, \text{ while } e^{\lambda_2 t} \to 0 \text{ as } t \to \infty \]

Call \( \lambda_2 \) the stable eigenvalue (\( v_2 \) the stable eigenvector) and \( \lambda_1 \) the unstable eigenvalue (\( v_1 \) the unstable eigenvector)

\[ z_2 = cz_1^{\lambda_2/\lambda_1}, \quad \lambda_2/\lambda_1 < 0 \]

Saddle
Phase Portrait of a Saddle Point
Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \quad \dot{z}_2 = \beta z_1 + \alpha z_2$$

$$r = \sqrt{z_1^2 + z_2^2}, \quad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

$$r(t) = r_0 e^{\alpha t} \quad \text{and} \quad \theta(t) = \theta_0 + \beta t$$

$$\alpha < 0 \quad \Rightarrow \quad r(t) \to 0 \text{ as } t \to \infty$$

$$\alpha > 0 \quad \Rightarrow \quad r(t) \to \infty \text{ as } t \to \infty$$

$$\alpha = 0 \quad \Rightarrow \quad r(t) \equiv r_0 \quad \forall \ t$$
\(\alpha < 0\)  
Stable Focus

\(\alpha > 0\)  
Unstable Focus

\(\alpha = 0\)  
Center
**Limit Cycles**

**Oscillation:** A system oscillates when it has a nontrivial periodic solution

\[ x(t + T) = x(t), \quad \forall \ t \geq 0 \]

**Linear (Harmonic) Oscillator:**

\[
\begin{bmatrix}
0 & -\beta \\
\beta & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
\]

\[ z_1(t) = r_0 \cos(\beta t + \theta_0), \quad z_2(t) = r_0 \sin(\beta t + \theta_0) \]

\[ r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \quad \theta_0 = \tan^{-1} \left[ \frac{z_2(0)}{z_1(0)} \right] \]
The linear oscillation is not practical because

- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation)

- The amplitude of oscillation depends on the initial conditions

The same problems exist with oscillation of nonlinear systems due to a center equilibrium point (e.g., pendulum without friction)
Limit Cycles:

Example: Negative Resistance Oscillator
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - \varepsilon h'(x_1)x_2
\end{align*}
\]

There is a unique equilibrium point at the origin.

\[
A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}
\]

\[
\lambda^2 + \varepsilon h'(0)\lambda + 1 = 0
\]

\[h'(0) < 0 \implies \text{Unstable Focus or Unstable Node}\]
Energy Analysis:

\[ E = \frac{1}{2} C v_C^2 + \frac{1}{2} L i_L^2 \]

\[ v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon} x_2 \]

\[ E = \frac{1}{2} C \left\{ x_1^2 + \left[ \varepsilon h(x_1) + x_2 \right]^2 \right\} \]

\[ \dot{E} = C \left\{ x_1 \dot{x}_1 + \left[ \varepsilon h(x_1) + x_2 \right] \left[ \varepsilon h'(x_1) \dot{x}_1 + \dot{x}_2 \right] \right\} \]

\[ = C \left\{ x_1 x_2 + \left[ \varepsilon h(x_1) + x_2 \right] \left[ \varepsilon h'(x_1) x_2 - x_1 - \varepsilon h'(x_1) x_2 \right] \right\} \]

\[ = C \left[ x_1 x_2 - \varepsilon x_1 h(x_1) - x_1 x_2 \right] \]

\[ = -\varepsilon C x_1 h(x_1) \]
\[
\dot{E} = -\varepsilon C x_1 h(x_1)
\]
Example: Van der Pol Oscillator

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon (1 - x_1^2)x_2 
\end{align*}
\]
Stable Limit Cycle

Unstable Limit Cycle