Chapter 14
Nonlinear Design Tools

Sliding Mode Control
Backstepping
Sliding Mode Control

Example

\[ \dot{x}_1 = x_2 \quad \dot{x}_2 = h(x) + g(x)u, \quad g(x) \geq g_0 > 0 \]

Sliding Manifold (Surface):

\[ s = a_1 x_1 + x_2 = 0 \]

\[ s(t) \equiv 0 \Rightarrow \dot{x}_1 = -a_1 x_1 \]

\[ a_1 > 0 \Rightarrow \lim_{t \to \infty} x_1(t) = 0 \]

How can we bring the trajectory to the manifold \( s = 0 \)?

How can we maintain it there?
\[ \dot{s} = a_1 \dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x)u \]

Suppose
\[ \left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \leq \varrho(x) \]

\[ V = \frac{1}{2} s^2 \]

\[ \dot{V} = s \dot{s} = s [a_1 x_2 + h(x)] + g(x)su \leq g(x)|s|\varrho(x) + g(x)su \]

\[ \beta(x) \geq \varrho(x) + \beta_0, \quad \beta_0 > 0 \]

\[ s > 0, \quad u = -\beta(x) \]

\[ \dot{V} \leq g(x)|s|\varrho(x) - g(x)\beta(x)|s| \leq g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s| \]
\[ s < 0, \quad u = \beta(x) \]

\[ \dot{V} \leq g(x)|s|\varphi(x) + g(x)su = g(x)|s|\varphi(x) - g(x)\beta(x)|s| \]

\[ \dot{V} \leq g(x)|s|\varphi(x) - g(x)(\varphi(x) + \beta_0)|s| = -g(x)\beta_0|s| \]

\[ \text{sgn}(s) = \begin{cases} 
1, & s > 0 \\
-1, & s < 0 
\end{cases} \]

\[ u = -\beta(x) \text{sgn}(s) \]

\[ \dot{V} \leq -g(x)\beta_0|s| \leq -g_0\beta_0|s| \]

\[ \dot{V} \leq -g_0\beta_0\sqrt{2V} \]
$$\dot{V} \leq -g_0\beta_0 \sqrt{2V}$$

$$\frac{dV}{\sqrt{V}} \leq -g_0\beta_0 \sqrt{2} \ dt$$

$$2 \sqrt{V} \left| \frac{V(s(t))}{V(s(0))} \right| \leq -g_0\beta_0 \sqrt{2} \ t$$

$$\sqrt{V(s(t))} \leq \sqrt{V(s(0))} - g_0\beta_0 \frac{1}{\sqrt{2}} \ t$$

$$|s(t)| \leq |s(0)| - g_0\beta_0 \ t$$

$s(t)$ reaches zero in finite time

Once on the surface $s = 0$, the trajectory cannot leave it.
What is the region of validity?
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= h(x) - g(x)\beta(x)\text{sgn}(s) \\
\dot{x}_1 &= -a_1x_1 + s \\
\dot{s} &= a_1x_2 + h(x) - g(x)\beta(x)\text{sgn}(s) \\
ss &\leq -g_0\beta_0|s|, \quad \text{if } \beta(x) \geq \varrho(x) + \beta_0 \\
V_1 &= \frac{1}{2}x_1^2 \\
\dot{V}_1 &= x_1\dot{x}_1 = -a_1x_1^2 + x_1s \leq -a_1x_1^2 + |x_1|c \leq 0 \\
\forall |s| &\leq c \text{ and } |x_1| \geq \frac{c}{a_1} \\
\Omega &= \left\{ |x_1| \leq \frac{c}{a_1}, |s| \leq c \right\} \\
\Omega \text{ is positively invariant if } \left| \frac{a_1x_2 + h(x)}{g(x)} \right| &\leq \varrho(x) \text{ over } \Omega
\end{align*}
\[ \Omega = \left\{ |x_1| \leq \frac{c}{a_1}, |s| \leq c \right\} \]

\[ \left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \leq k_1 < k, \quad \forall \ x \in \Omega \]

\[ u = -k \ \text{sgn}(s) \]
Chattering

How can we reduce or eliminate chattering?
pendulum equation

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(g_0/\ell) \sin(x_1 + \delta_1) - (k_0/m)x_2 + (1/ml^2)u \\
u &= -k \operatorname{sgn}(s) = -k \operatorname{sgn}(a_1 x_1 + x_2)
\end{align*}
\]

to stabilize the pendulum at \( \delta_1 = \pi/2 \), where \( x_1 = \theta - \delta_1 \) and \( x_2 = \dot{\theta} \). The constants \( m, \ell, k_0, \) and \( g_0 \) are the mass, length, coefficient of friction, and acceleration due to gravity, respectively. We take \( a_1 = 1 \) and \( k = 4 \). The gain \( k = 4 \) is chosen by using

\[
\left| \frac{a_1 x_2 + h(x)}{g} \right| = \left| \ell^2 (m - k_0)x_2 - mg_0 \ell \cos(x_1) \right|
\leq \ell^2 |m - k_0|(2\pi) + mg_0 \ell \leq 3.68
\]

where the bound is calculated over the set \( \{|x_1| \leq \pi, \ |x_1 + x_2| \leq \pi\} \) for \( 0.05 \leq m \leq 0.2, \ 0.9 \leq \ell \leq 1.1, \) and \( 0 \leq k_0 \leq 0.05 \). The simulation is performed by using \( m = 0.1, \ \ell = 1, \) and \( k_0 = 0.02 \). Figure 14.4 shows ideal sliding mode control, while Figure 14.5 shows a nonideal case where switching is delayed by unmodeled actuator dynamics having the transfer function \( 1/(0.01s + 1)^2 \).
Figure 14.4: Ideal sliding mode control.
Figure 14.5: Sliding mode control with unmodeled actuator dynamics.
Reduce the amplitude of the signum function

\[ \dot{s} = a_1 x_2 + h(x) + g(x)u \]

\[ u = -\frac{[a_1 x_2 + \hat{h}(x)]}{\hat{g}(x)} + v \]

\[ \dot{s} = \delta(x) + g(x)v \]

\[ \delta(x) = a_1 \left[ 1 - \frac{g(x)}{\hat{g}(x)} \right] x_2 + h(x) - \frac{g(x)}{\hat{g}(x)} \hat{h}(x) \]

\[ \left| \frac{\delta(x)}{g(x)} \right| \leq \varrho(x), \quad \beta(x) \geq \varrho(x) + \beta_0 \]

\[ v = -\beta(x) \text{ sgn}(s) \]
amplitude of the switching component would be smaller. For example, returning to the pendulum equation and taking $\hat{m} = 0.125$, $\hat{l} = 1$, $\hat{k}_0 = 0.025$ to be nominal values of $m$, $l$, $k_0$, we have

$$\left| \frac{\delta(x)}{g} \right| = \left| \left( a_1 ml^2 - a_1 \hat{m} \hat{l}^2 - k_0 l^2 + \hat{k}_0 \hat{l}^2 \right) x_2 - g_0 (ml - \hat{m} \hat{l}) \cos x_1 \right| \leq 1.83$$

where the bound is calculated over the same set as before. The modified sliding mode control is taken as

$$u = -0.1 x_2 + 1.2263 \cos x_1 - 2 \text{ sgn}(s)$$

which shows a reduction in the switching term amplitude from 4 to 2. Figure 14.6 shows simulation of this modified control in the presence of unmodeled actuator dynamics. The reduction in the amplitude of chattering is clear.
Figure 14.6: Modified sliding mode control with unmodeled actuator dynamics.
Replace the signum function by a high-slope saturation function

\[ u = -\beta(x) \, \text{sat} \left( \frac{s}{\varepsilon} \right) \]

\[ \text{sat}(y) = \begin{cases} 
  y, & \text{if } |y| \leq 1 \\
  \text{sgn}(y), & \text{if } |y| > 1 
\end{cases} \]
How can we analyze the system?

For $|s| \geq \varepsilon$, $u = -\beta(x) \text{sgn}(s)$

With $c \geq \varepsilon$

- $\Omega = \left\{ \left| x_1 \right| \leq \frac{c}{a_1}, \ |s| \leq c \right\}$ is positively invariant

- The trajectory reaches the boundary layer $\{ |s| \leq \varepsilon \}$ in finite time

- The boundary layer is positively invariant
Inside the boundary layer:

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + s \\
\dot{s} &= a_1 x_2 + h(x) - g(x) \beta(x) \frac{s}{\varepsilon}
\end{align*}
\]

\[
x_1 \dot{x}_1 \leq -a_1 x_1^2 + |x_1| \varepsilon
\]

\[
0 < \theta < 1
\]

\[
x_1 \dot{x}_1 \leq -(1 - \theta) a_1 x_1^2, \quad \forall |x_1| \geq \frac{\varepsilon}{\theta a_1}
\]

The trajectories reach the positively invariant set

\[
\Omega_{\varepsilon} = \{ |x_1| \leq \frac{\varepsilon}{\theta a_1}, \ |s| \leq \varepsilon \}
\]

in finite time
\( \varepsilon \}) in finite time. In general, we do not stabilize the origin, but we achieve ultimate boundedness with an ultimate bound that can be reduced by decreasing \( \varepsilon \). What happens inside \( \Omega_\varepsilon \) is problem dependent. Let us consider again the pendulum equation and see what happens inside \( \Omega_\varepsilon \) in that case. Inside the boundary layer \( \{|s| \leq \varepsilon\} \), the control reduces to the linear feedback law \( u = -ks/\varepsilon \), and the closed-loop system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(g_0/\ell)\sin(x_1 + \delta_1) - (k_0/m)x_2 - (k/ml^2\varepsilon)(a_1x_1 + x_2)
\end{align*}
\]

has a unique equilibrium point at \( (\bar{x}_1, 0) \), where \( \bar{x}_1 \) satisfies the equation

\[
\varepsilon mg_0 \ell \sin(\bar{x}_1 + \delta_1) + ka_1 \bar{x}_1 = 0
\]

and can be approximated for small \( \varepsilon \) by \( \bar{x}_1 \approx -(\varepsilon mg_0 \ell/ka_1) \sin \delta_1 \). Shifting the
Figure 14.8: "Continuous" sliding mode control.
Figure 14.9: “Continuous” sliding mode control with unmodeled actuator dynamics.
Stabilization by Sliding Mode Control

Regular Form:

\[
\begin{align*}
\dot{\eta} &= f_a(\eta, \xi) \\
\dot{\xi} &= f_b(\eta, \xi) + g(\eta, \xi)u + \delta(t, \eta, \xi, u)
\end{align*}
\]

\[\eta \in \mathbb{R}^{n-1}, \ \xi \in \mathbb{R}, \ u \in \mathbb{R}\]

\[f_a(0, 0) = 0, \ f_b(0, 0) = 0, \ g(\eta, \xi) \geq g_0 > 0\]

Sliding Manifold:

\[s = \xi - \phi(\eta) = 0, \ \phi(0) = 0\]

\[s(t) \equiv 0 \ \Rightarrow \ \dot{\eta} = f_a(\eta, \phi(\eta))\]

Design \(\phi\) s.t. the origin of \(\dot{\eta} = f_a(\eta, \phi(\eta))\) is asymptotically stable.
$\dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + g(\eta, \xi)u + \delta(t, \eta, \xi, u)$

$u = -\frac{1}{\hat{g}} \left( \hat{f}_b - \frac{\partial \phi}{\partial \eta} \hat{f}_a \right) + v \quad \text{or} \quad u = v$

$u = -L \left( \hat{f}_b - \frac{\partial \phi}{\partial \eta} \hat{f}_a \right) + v, \quad L = \frac{1}{\hat{g}} \quad \text{or} \quad L = 0$

$\dot{s} = g(\eta, \xi) v + \Delta(t, \eta, \xi, v)$

$\Delta = f_b - \frac{\partial \phi}{\partial \eta} f_a + \delta - gL \left( \hat{f}_b - \frac{\partial \phi}{\partial \eta} \hat{f}_a \right)$

$\left| \frac{\Delta(t, \eta, \xi, v)}{g(\eta, \xi)} \right| \leq g(\eta, \xi) + \kappa_0 |v|$
\[
\left| \frac{\Delta(t, \eta, \xi, v)}{g(\eta, \xi)} \right| \leq \varrho(\eta, \xi) + \kappa_0 |v|
\]

\( \varrho(\eta, \xi) \geq 0, \ 0 \leq \kappa_0 < 1 \) (\textit{Known})

\[
s \dot{s} = s g v + s \Delta \leq s g v + |s| |\Delta|
\]

\[
s \dot{s} \leq g [s v + |s| (\varrho + \kappa_0 |v|)]
\]

\[
v = -\beta(\eta, \xi) \text{sgn}(s)
\]

\[
\beta(\eta, \xi) \geq \frac{\varrho(\eta, \xi)}{1 - \kappa_0} + \beta_0, \ \beta_0 > 0
\]

\[
s \dot{s} \leq g [-\beta |s| + \varrho |s| + \kappa_0 \beta |s|] = g [-\beta (1 - \kappa_0)|s| + \varrho |s|] \]

\[
s \dot{s} \leq g [-\varrho |s| - (1 - \kappa_0) \beta_0 |s| + \varrho |s|]
\]
\[ s \dot{s} \leq -g(\eta, \xi)(1 - \kappa_0)\beta_0 |s| \leq -g_0\beta_0(1 - \kappa_0)|s| \]

\[ \nu = -\beta(x) \text{sat} \left( \frac{s}{\varepsilon} \right), \quad \varepsilon > 0 \]

\[ s \dot{s} \leq -g_0\beta_0(1 - \kappa_0)|s|, \quad \text{for } |s| \geq \varepsilon \]

The trajectory reaches the boundary layer \{ |s| \leq \varepsilon \} in finite time and remains inside thereafter.

Study the behavior of \( \eta \)

\[ \dot{\eta} = f_a(\eta, \phi(\eta) + s) \]

What do we know about this system and what do we need?
\[ \alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|) \]

\[ \frac{\partial V}{\partial \eta} f_a(\eta, \phi(\eta) + s) \leq -\alpha_3(\|\eta\|), \quad \forall \|\eta\| \geq \gamma(|s|) \]

\[ |s| \leq c \Rightarrow \dot{V} \leq -\alpha_3(\|\eta\|), \text{ for } \|\eta\| \geq \gamma(c) \]

\[ \alpha(r) = \alpha_2(\gamma(r)) \]

\[ V(\eta) \geq \alpha(c) \Leftrightarrow V(\eta) \geq \alpha_2(\gamma(c)) \Rightarrow \alpha_2(\|\eta\|) \geq \alpha_2(\gamma(c)) \]

\[ \Rightarrow \|\eta\| \geq \gamma(c) \Rightarrow \dot{V} \leq -\alpha_3(\|\eta\|) \leq -\alpha_3(\gamma(c)) \]

The set \( \{V(\eta) \leq c_0\} \) with \( c_0 \geq \alpha(c) \) is positively invariant

\[ \Omega = \{V(\eta) \leq c_0\} \times \{|s| \leq c\}, \text{ with } c_0 \geq \alpha(c) \]
\[ \Omega = \{ V(\eta) \leq c_0 \} \times \{|s| \leq c\}, \text{ with } c_0 \geq \alpha(c) \]

is positively invariant and all trajectories starting in \( \Omega \) reach 
\[ \Omega_\varepsilon = \{ V(\eta) \leq \alpha(\varepsilon) \} \times \{|s| \leq \varepsilon\} \] in finite time
Theorem 14.1: Suppose all the assumptions hold over $\Omega$. Then, for all $(\eta(0), \xi(0)) \in \Omega$, the trajectory $(\eta(t), \xi(t))$ is bounded for all $t \geq 0$ and reaches the positively invariant set $\Omega_{\varepsilon}$ in finite time. If the assumptions hold globally and $V(\eta)$ is radially unbounded, the foregoing conclusion holds for any initial state.

Theorem 14.2: Suppose all the assumptions hold over $\Omega$

- $\rho(0) = 0$, $\kappa_0 = 0$
- The origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is exponentially stable

Then there exits $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the origin of the closed-loop system is exponentially stable and $\Omega$ is a subset of its region of attraction. If the assumptions hold globally, the origin will be globally uniformly asymptotically stable.
Example

\[ \dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \quad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u \]

\[ |\theta_1| \leq a, \quad |\theta_2| \leq b \]

\[ x_2 = -kx_1 \quad \Rightarrow \quad \dot{x}_1 = -kx_1 + \theta_1 x_1 \sin x_2 \]

\[ V_1 = \frac{1}{2} x_1^2 \quad \Rightarrow \quad x_1 \dot{x}_1 \leq -kx_1^2 + ax_1^2 \]

\[ s = x_2 + kx_1, \quad k > a \]

\[ \dot{s} = \theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_2) \]

\[ u = -x_1 - kx_2 + v \quad \Rightarrow \quad \dot{s} = v + \Delta(x) \]

\[ \Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2 \]
$$\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$$

$$|\Delta(x)| \leq ak|x_1| + bx_2^2$$

$$\beta(x) = ak|x_1| + bx_2^2 + \beta_0, \quad \beta_0 > 0$$

$$u = -x_1 - kx_2 - \beta(x) \operatorname{sgn}(s)$$
Backstepping

\[ \dot{\eta} = f(\eta) + g(\eta)\xi \]
\[ \dot{\xi} = u, \quad \eta \in \mathbb{R}^n, \; \xi, \; u \in \mathbb{R} \]

Stabilize the origin using state feedback

View \( \xi \) as “virtual” control input to

\[ \dot{\eta} = f(\eta) + g(\eta)\xi \]

Suppose there is \( \xi = \phi(\eta) \) that stabilizes the origin of

\[ \dot{\eta} = f(\eta) + g(\eta)\phi(\eta) \]

\[ \frac{\partial V}{\partial \eta} [f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \quad \forall \; \eta \in D \]
\[ z = \dot{\xi} - \phi(\eta) \]

\[ \dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \]

\[ \dot{z} = u - \frac{\partial\phi}{\partial\eta}[f(\eta) + g(\eta)\xi] \]

\[ u = \frac{\partial\phi}{\partial\eta}[f(\eta) + g(\eta)\xi] + v \]

\[ \dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \]

\[ \dot{z} = v \]
\[ V_c(\eta, \xi) = V(\eta) + \frac{1}{2} z^2 \]

\[
\dot{V}_c = \frac{\partial V}{\partial \eta} [f(\eta) + g(\eta) \phi(\eta)] + \frac{\partial V}{\partial \eta} g(\eta) z + zv
\]

\[
\leq -W(\eta) + \frac{\partial V}{\partial \eta} g(\eta) z + zv
\]

\[ v = -\frac{\partial V}{\partial \eta} g(\eta) - k z, \quad k > 0 \]

\[ \dot{V}_c \leq -W(\eta) - k z^2 \]

\[ u = \frac{\partial \phi}{\partial \eta} [f(\eta) + g(\eta) \xi] - \frac{\partial V}{\partial \eta} g(\eta) - k [\xi - \phi(\eta)] \]
Example

\[ \dot{x}_1 = x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = u \]

\[ \dot{x}_1 = x_1^2 - x_1^3 + x_2 \]

\[ x_2 = \phi(x_1) = -x_1^2 - x_1 \quad \Rightarrow \quad \dot{x}_1 = -x_1 - x_1^3 \]

\[ V(x_1) = \frac{1}{2}x_1^2 \quad \Rightarrow \quad \dot{V} = -x_1^2 - x_1^4, \quad \forall \ x_1 \in \mathbb{R} \]

\[ z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2 \]

\[ \dot{x}_1 = -x_1 - x_1^3 + z_2 \]

\[ \dot{z}_2 = u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \]
\[ V_c(x) = \frac{1}{2} x_1^2 + \frac{1}{2} z_2^2 \]

\[
\dot{V}_c = x_1 (-x_1 - x_1^3 + z_2) \\
+ z_2 [u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)]
\]

\[
\dot{V}_c = -x_1^2 - x_1^4 \\
+ z_2 [x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u]
\]

\[ u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \]

\[ \dot{V}_c = -x_1^2 - x_1^4 - z_2^2 \]

\[ \dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 \]
Example

\[
\begin{align*}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u \\
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = x_3 \\
x_3 &= -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \overset{\text{def}}{=} \phi(x_1, x_2) \\
V(x) &= \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2, \quad \dot{V} = -x_1^2 - x_1^4 - z_2^2 \\
z_3 &= x_3 - \phi(x_1, x_2) \\
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = \phi(x_1, x_2) + z_3 \\
\dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3)
\end{align*}
\]
\[ V_c = V + \frac{1}{2} z_3^2 \]

\[ \dot{V}_c = \frac{\partial V}{\partial x_1} (x_1^2 - x_1^3 + x_2) + \frac{\partial V}{\partial x_2} (z_3 + \phi) + z_3 \left[ u - \frac{\partial \phi}{\partial x_1} (x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2} (z_3 + \phi) \right] \]

\[ \dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 + z_3 \left[ \frac{\partial V}{\partial x_2} - \frac{\partial \phi}{\partial x_1} (x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2} (z_3 + \phi) + u \right] \]

\[ u = -\frac{\partial V}{\partial x_2} + \frac{\partial \phi}{\partial x_1} (x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2} (z_3 + \phi) - z_3 \]
\[ \dot{\eta} = f(\eta) + g(\eta)\xi \]
\[ \dot{\xi} = f_a(\eta, \xi) + g_a(\eta, \xi)u, \quad g_a(\eta, \xi) \neq 0 \]

\[ u = \frac{1}{g_a(\eta, \xi)}[v - f_a(\eta, \xi)] \]

\[ \dot{\eta} = f(\eta) + g(\eta)\xi \]
\[ \dot{\xi} = v \]

\[ u = \phi_c(\eta, \xi) \]
\[ = \frac{1}{g_a(\eta, \xi)} \left\{ \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta}g(\eta) - k[\xi - \phi(\eta)] - f_a(\eta, \xi) \right\} \]

\[ V_c(\eta, \xi) = V(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^2 \]
Strict-Feedback Form

\[
\begin{align*}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u \\
\end{align*}
\]

\[g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k\]
over the domain of interest. The recursive procedure starts with the system

$$
\dot{x} = f_0(x) + g_0(x)z_1
$$

where $z_1$ is viewed as the control input. We assume that it is possible to determine a stabilizing state feedback control law $z_1 = \phi_0(x)$, with $\phi_0(0) = 0$, and a Lyapunov function $V_0(x)$ such that

$$
\frac{\partial V_0}{\partial x} [f_0(x) + g_0(x)\phi_0(x)] \leq -W(x)
$$

over the domain of interest for some positive definite function $W(x)$. In many applications of backstepping, the variable $x$ is scalar, which simplifies this stabilization problem. With $\phi_0(x)$ and $V_0(x)$ in hand, we proceed to apply backstepping in a systematic way. First, we consider the system

$$
\begin{align*}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2
\end{align*}
$$

as a special case of (14.53)–(14.54) with

$$
\eta = x, \quad \xi = z_1, \quad u = z_2, \quad f = f_0, \quad g = g_0, \quad f_a = f_1, \quad g_a = g_1
$$
We use (14.56) and (14.57) to obtain the stabilizing state feedback control law and the Lyapunov function as

\[
\phi_1(x, z_1) = \frac{1}{g_1} \left[ \frac{\partial \phi_0}{\partial x} (f_0 + g_0 z_1) - \frac{\partial V_0}{\partial x} g_0 - k_1 (z_1 - \phi) - f_1 \right], \quad k_1 > 0
\]

\[
V_1(x, z_1) = V_0(x) + \frac{1}{2} [z_1 - \phi(x)]^2
\]

Next, we consider the system

\[
\begin{align*}
\dot{x} &= f_0(x) + g_0(x) z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3
\end{align*}
\]

as a special case of (14.53)–(14.54) with

\[
\eta = \begin{bmatrix} x \\ z_1 \end{bmatrix}, \quad \xi = z_2, \quad u = z_3, \quad f = \begin{bmatrix} f_0 + g_0 z_1 \\ f_1 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}, \quad f_a = f_2, \quad g_a = g_2
\]
Using (14.56) and (14.57), we obtain the stabilizing state feedback control law and the Lyapunov function as

$$\phi_2(x, z_1, z_2) = \frac{1}{g_2} \left[ \frac{\partial \phi_1}{\partial x}(f_0 + g_0 z_1) + \frac{\partial \phi_1}{\partial z_1}(f_1 + g_1 z_2) - \frac{\partial V_1}{\partial z_1} g_1 - k_2(z_2 - \phi_1) - f_2 \right]$$

for some $k_2 > 0$ and

$$V_2(x, z_1, z_2) = V_1(x, z_1) + \frac{1}{2}[z_2 - \phi_2(x, z_1)]^2$$

This process is repeated $k$ times to obtain the overall stabilizing state feedback control law $u = \phi_k(x, z_1, \ldots, z_k)$ and the Lyapunov function $V_k(x, z_1, \ldots, z_k)$. 