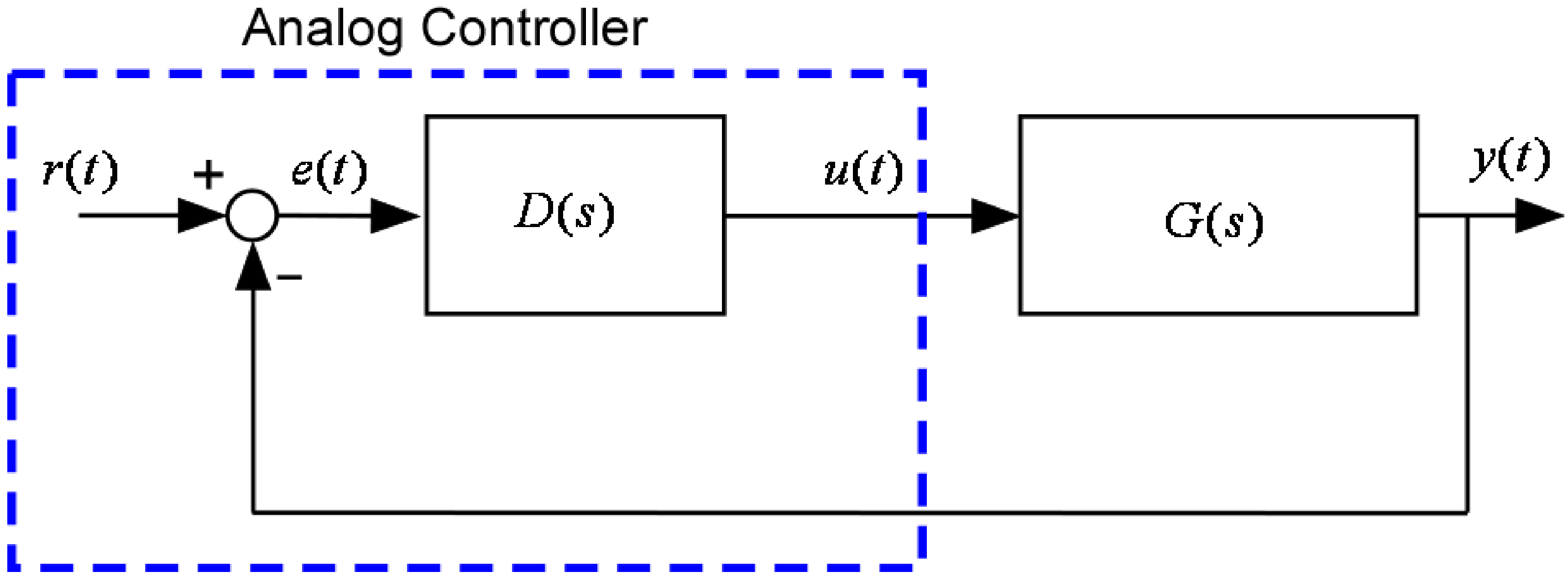
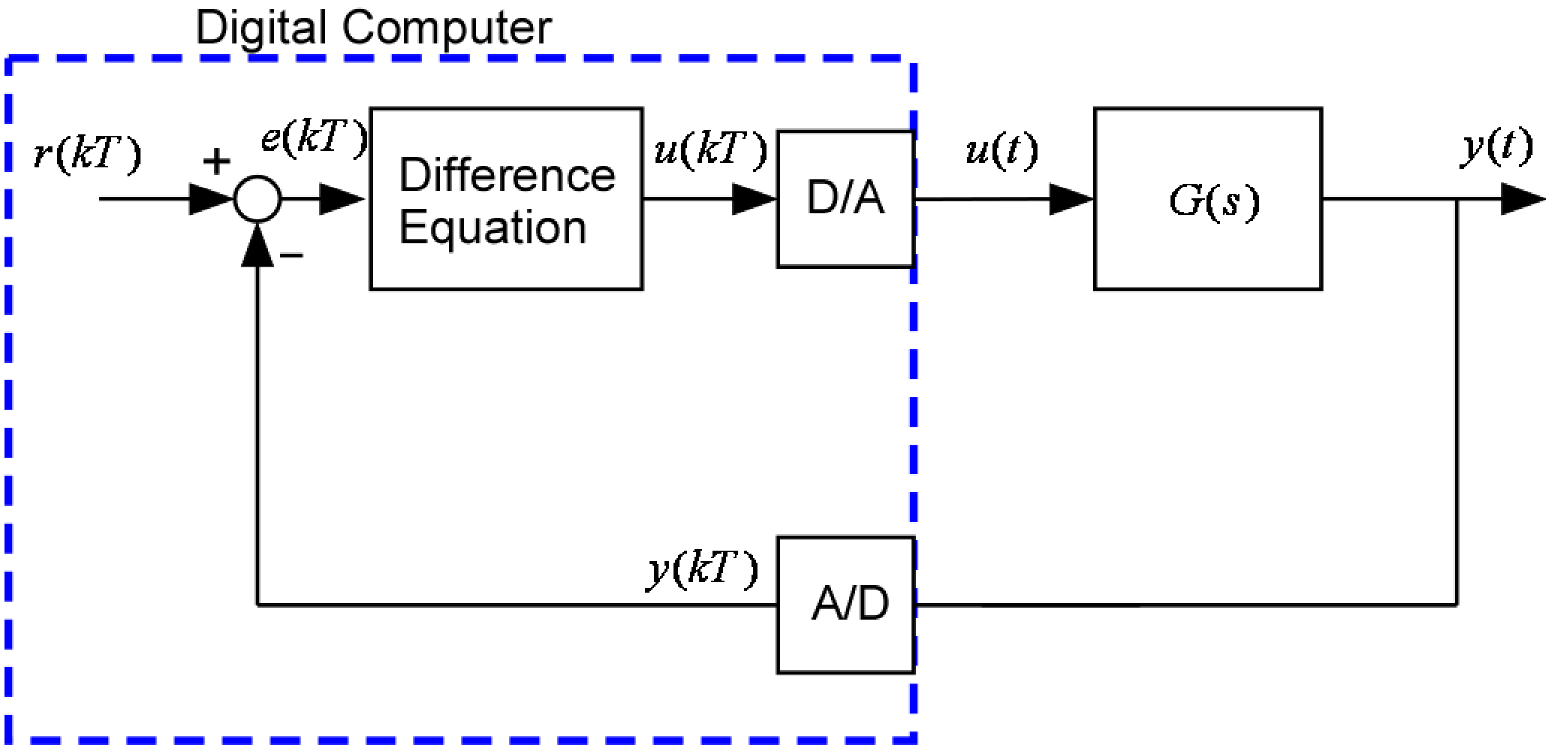
Discrete Models of Sampled-Data Systems

Digital Control Systems

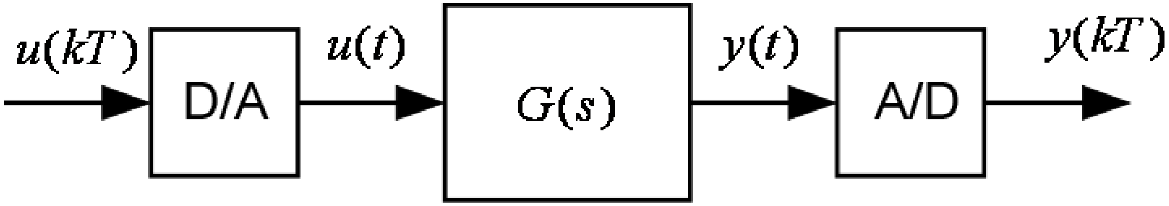
Continuous-Time Control Systems

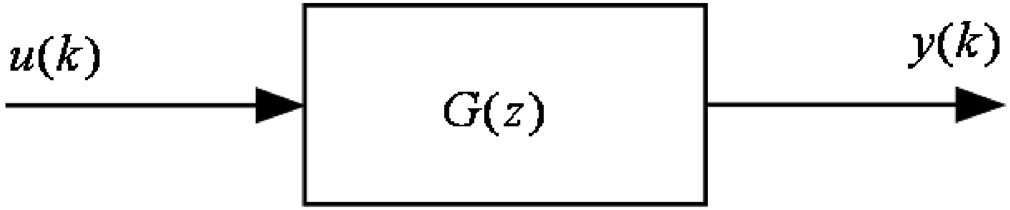


Digital Control Systems



Sampled-Data Systems





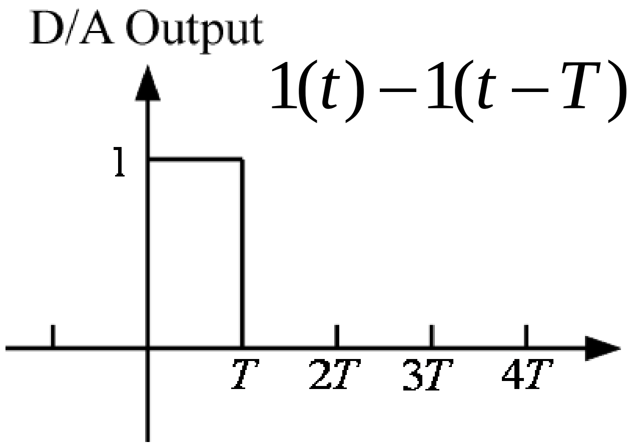
Discrete Transfer Function

$$G(z) = \frac{Y(z)}{U(z)}$$

Unit Pulse: $e(k) = \begin{cases} 1, k = 0 \\ 0, k \neq 0 \end{cases} = \delta(k)$

$$\mathcal{Z}[\delta(k)] = \sum_{k=0}^{\infty} e(k) z^{-k} = 1$$

$$U(z) = 1, Y(z) = G(z)U(z) = G(z)$$



$$\mathcal{L}[1(t) - 1(t-T)] = \frac{1}{s} - \frac{e^{-sT}}{s}$$

Discrete Transfer Function

$$Y(s) = (1 - e^{-Ts})\frac{G(s)}{s}$$

$$G(z) = \mathcal{Z}[Y(kT)] = \mathcal{Z}[\mathcal{L}^{-1}\{Y(s)\}] \triangleq \mathcal{Z}[Y(s)]$$

$$= \mathcal{Z}\left[(1 - e^{-Ts})\frac{G(s)}{s}\right] = \mathcal{Z}\left[\frac{G(s)}{s}\right] - \mathcal{Z}\left[e^{-Ts}\frac{G(s)}{s}\right]$$

$$\mathcal{Z}\left[e^{-Ts}\frac{G(s)}{s}\right] = z^{-1}\mathcal{Z}\left[\frac{G(s)}{s}\right]$$

$$G(z) = (1 - z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right]$$

Discrete Transfer Function: Example

$$G(s)=\frac{a}{s+a},\frac{G(s)}{s}=\frac{a}{s(s+a)}=\frac{1}{s}-\frac{1}{s+a}$$

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{s}\right\}=1(t)-e^{-at}1(t),1(kT)-e^{-akT}1(kT)$$

$$\mathscr{Z}\left[\frac{G(s)}{s}\right]=\frac{z}{z-1}-\frac{z}{z-e^{-aT}}=\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

$$G(z)=(1-z^{-1})\mathscr{Z}\left[\frac{G(s)}{s}\right]=\frac{z-1}{z}\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

$$=\frac{(1-e^{-aT})}{(z-e^{-aT})}$$

Discrete Transfer Function: Example

$$G(s) = \frac{1}{s^2}$$

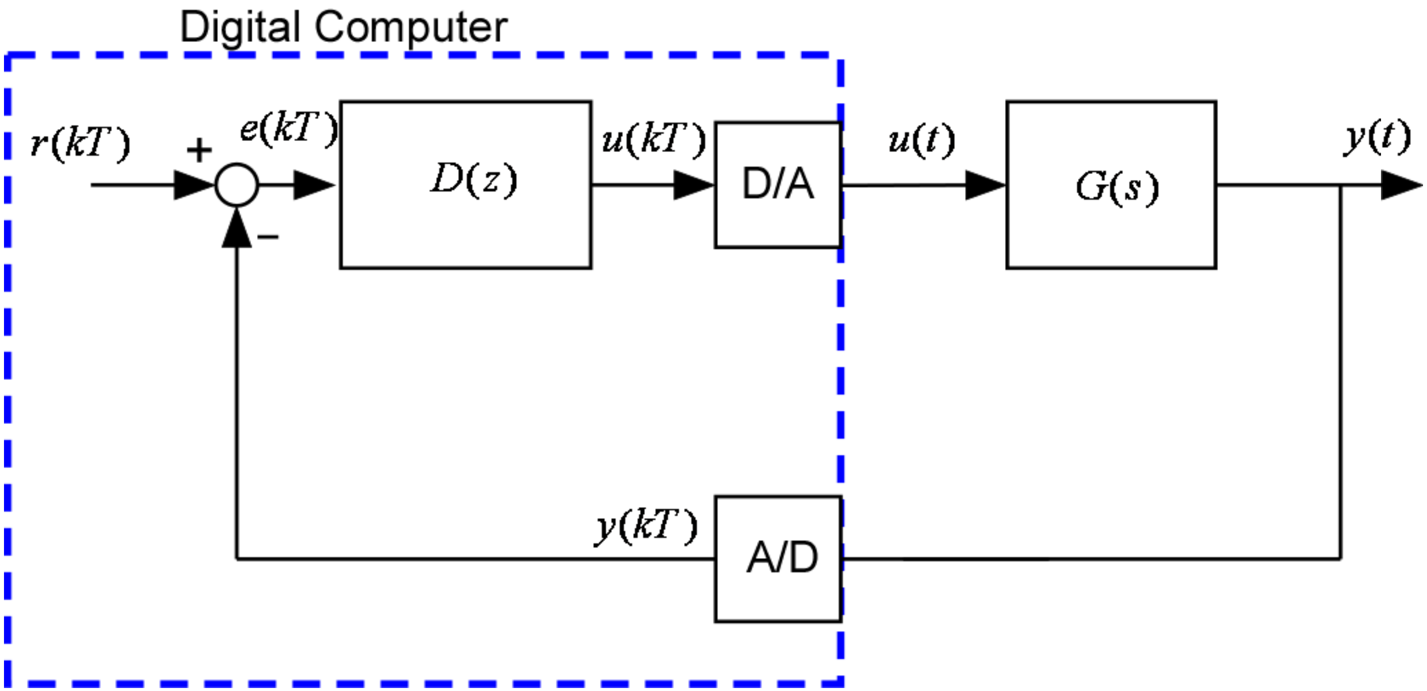
$$G(z) = (1 - z^{-1})\mathcal{Z}\left[\frac{1}{s^3}\right] = \frac{z-1}{z}\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2}\frac{(z+1)}{(z-1)^2}$$

MATLAB

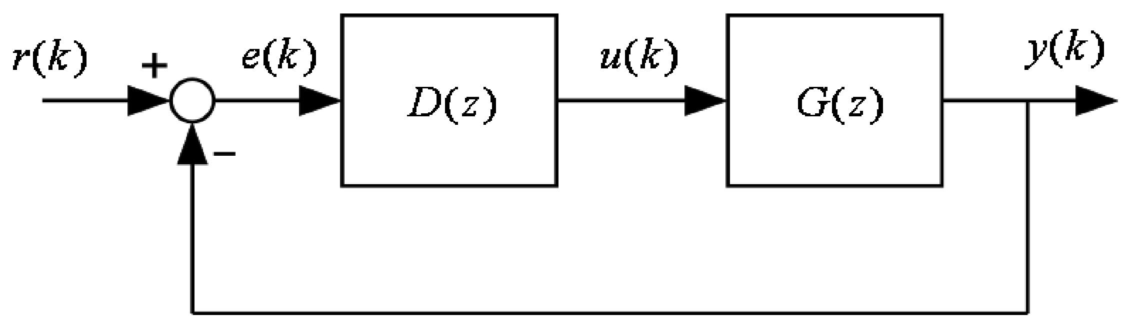
```
T=1
numC=1,denC=[1 0 0]
sysC=tf(numC,denC)
sysD=c2d(sysC,T)
[numD,denD,T]=tfdata(sysD)
```

numD=[0 0.5 0.5] and denD=[1 -2 1]

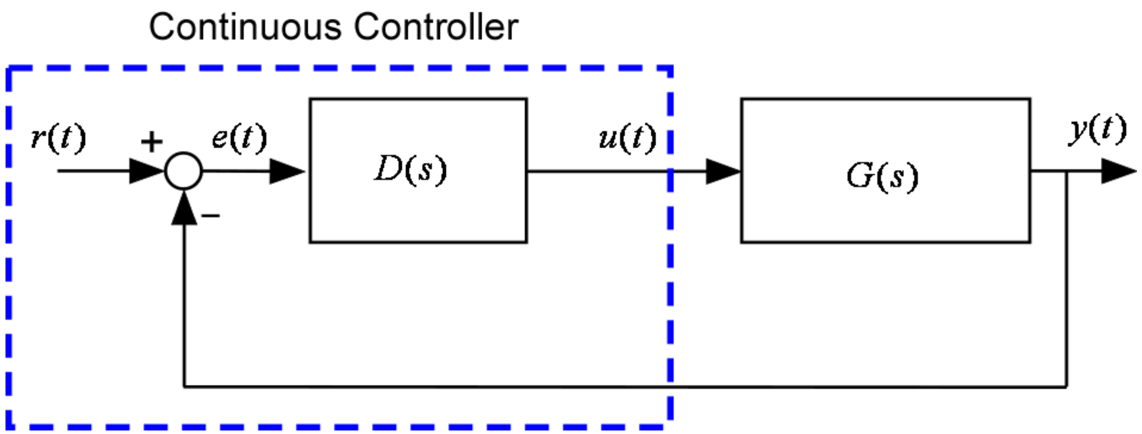
Closed-Loop Digital Control System



$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + D(z)G(z)}$$



Closed-Loop Control Systems



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + D(s)G(s)}$$

$$\text{2nd order sysetm:} \frac{Y(s)}{R(s)} = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Mapping s-plane into z-plane

$$z = e^{sT}\Big|_{s=s_{1,2}} = e^{-\zeta\omega_n T} e^{\pm j\omega_n T\sqrt{1-\zeta^2}} = e^{-\zeta\omega_n T}\measuredangle\left(\pm\omega_n T\sqrt{1-\zeta^2}\right) = r\measuredangle(\pm\theta)$$

$$\left.\begin{aligned} e^{-\zeta\omega_n T} &= r \rightarrow \zeta\omega_n T = -\ln r \\ \omega_n T\sqrt{1-\zeta^2} &= \theta \end{aligned}\right\} \rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{-\ln r}{\theta}$$

$$\zeta = \frac{-\ln r}{\sqrt{\ln^2 r + \theta^2}}, \omega_n = \frac{1}{T}\sqrt{\ln^2 r + \theta^2}$$

$$\tau = \frac{1}{\zeta\omega_n} = \frac{T}{\ln r}$$

Example

$$G(s) = \frac{1}{s(s+1)}, D(s) = 1$$

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + D(s)G(s)} = \frac{1}{s^2 + s + 1}$$

$$\omega_n = 1, \zeta = 0.5, \tau = 2$$

Overshoot: 18%

$$T = 1$$

$$G(z) = \frac{z-1}{z} \mathcal{Z}\left[\frac{1}{s^2(s+1)}\right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

$$\frac{G(z)}{1 + G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

Example

$$\frac{G(z)}{1+G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

$$z^2 - z + 0.632 = 0 \rightarrow z = 0.5 \pm j0.618 = 0.795\measuredangle(\pm 51.0^\circ) = 0.795\measuredangle(\pm 0.890 rad)$$

$$r = 0.795, \theta = 0.890$$

$$\zeta = \frac{-\ln(0.795)}{\sqrt{\ln^2 0.795 + 0.890^2}} = 0.250, \omega_n = \frac{1}{1}\sqrt{\ln^2 0.795 + 0.890^2} = 0.9191$$

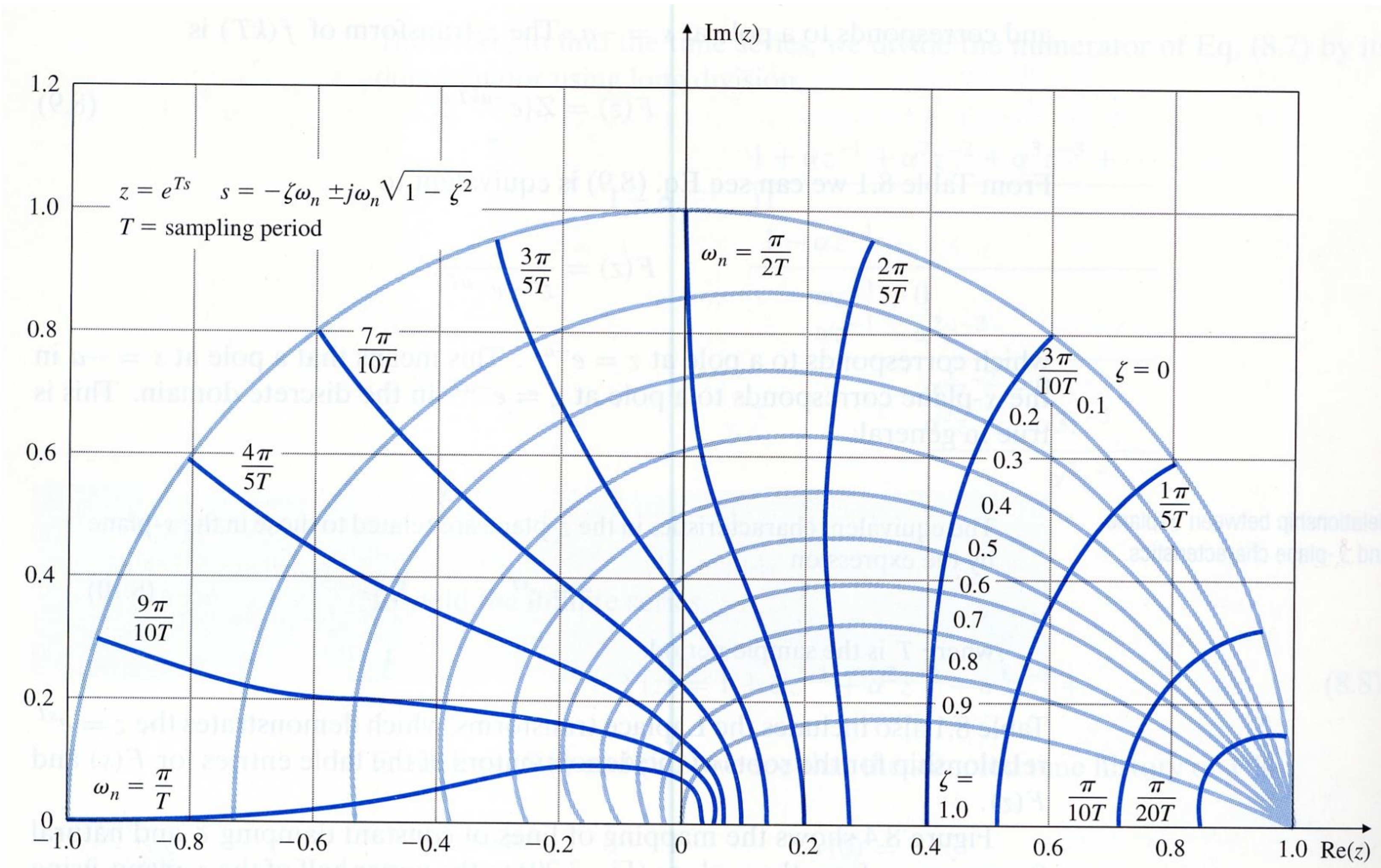
$$\tau = \frac{-1}{\ln(0.795)} = 4.36$$

Example: MATLAB

```
>> T=1;
>> numC=1;denC=[1 1 0];
>> sysC=tf(numC,denC);
>> sysD=c2d(sysC,T)

Transfer function:
  0.3679 z + 0.2642
----------------------
z^2 - 1.368 z + 0.3679

Sampling time: 1
>> clsysD=feedback(sysD,1)

Transfer function:
0.3679 z + 0.2642
-----------------
z^2 - z + 0.6321

Sampling time: 1
```

Example: MATLAB

```
>> [clnumD,cldenD,T]=tfdata(clsysD,'v')

clnumD =

         0  3.6788e-001  2.6424e-001


cldenD =

  1.0000e+000 -1.0000e+000  6.3212e-001


T =

     1

>> roots(cldenD)

ans =

  5.0000e-001 +6.1816e-001i
  5.0000e-001 -6.1816e-001i
```



Im(z)
Re(z)
1.2
1.0
0.8
0.6
0.4
0.2
0
−1.0
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1.0
$z = e^{Ts}$ $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
T = sampling period
$\omega_n = \frac{\pi}{2T}$
$\frac{3\pi}{5T}$
$\frac{2\pi}{5T}$
$\frac{7\pi}{10T}$
$\frac{3\pi}{10T}$
$\zeta = 0$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\zeta = 1.0$
$\frac{4\pi}{5T}$
$\frac{1\pi}{5T}$
$\frac{9\pi}{10T}$
$\omega_n = \frac{\pi}{T}$
$\frac{\pi}{10T}$
$\frac{\pi}{20T}$

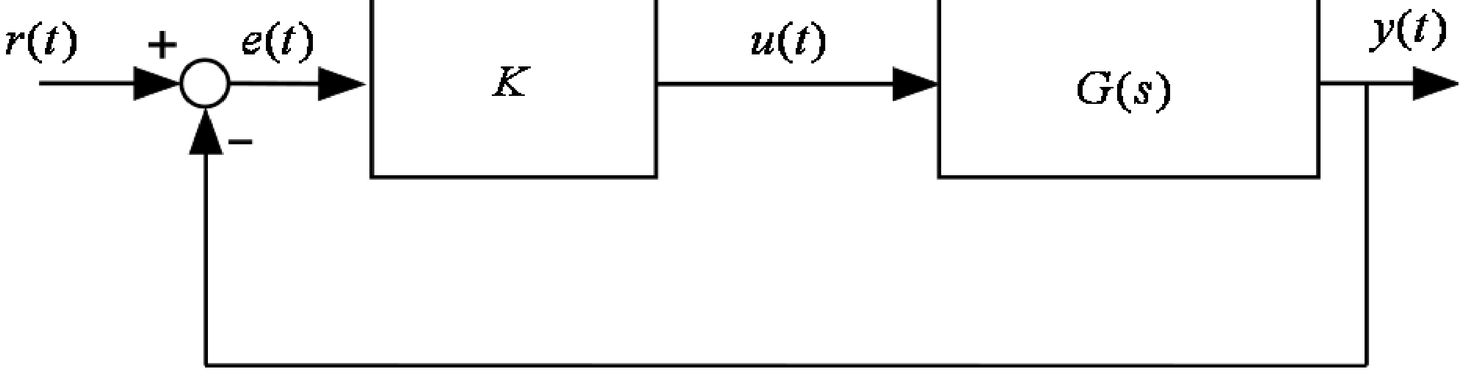
Example

$$T = 0.1$$

$$\zeta = 0.475, \omega_n = 0.998, \tau = 4.36$$

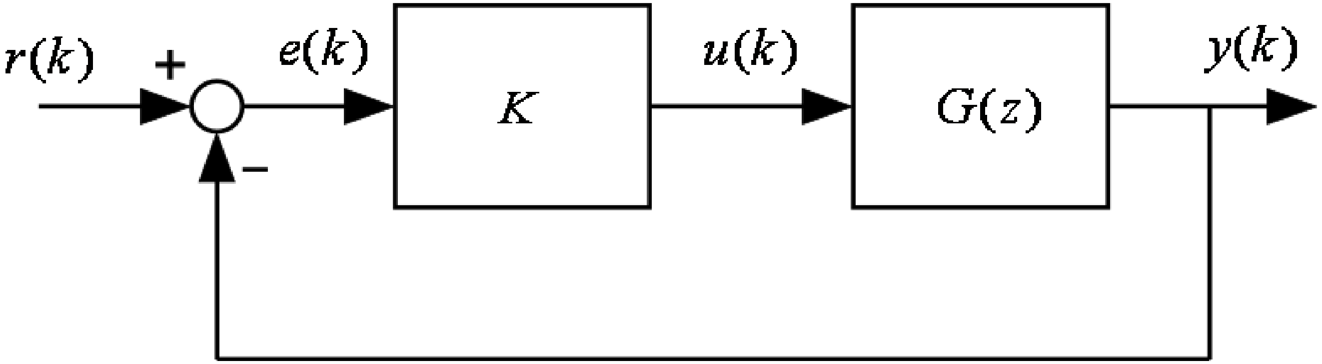
```
>> T=0.1;
>> sysD=c2d(sysC,T)

Transfer function:
0.004837 z + 0.004679
----------------------
z^2 - 1.905 z + 0.9048
Sampling time: 0.1
.
.
.
.
>> roots(cldenD)

ans =

  9.5000e-001 +8.3763e-002i
  9.5000e-001 -8.3763e-002i
```

Root Locus



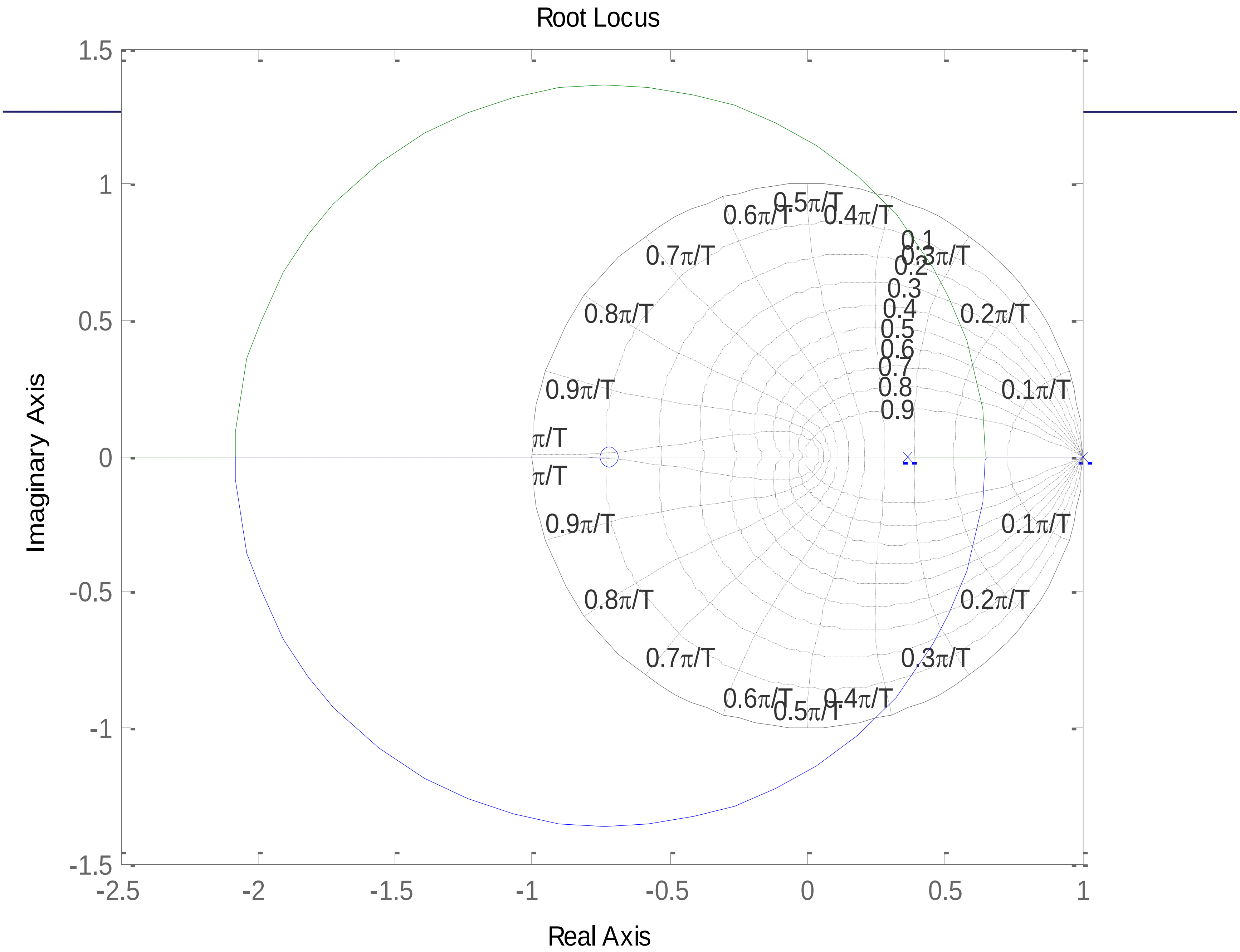
$$1 + KG(s) = 0$$

Discrete Root Locus

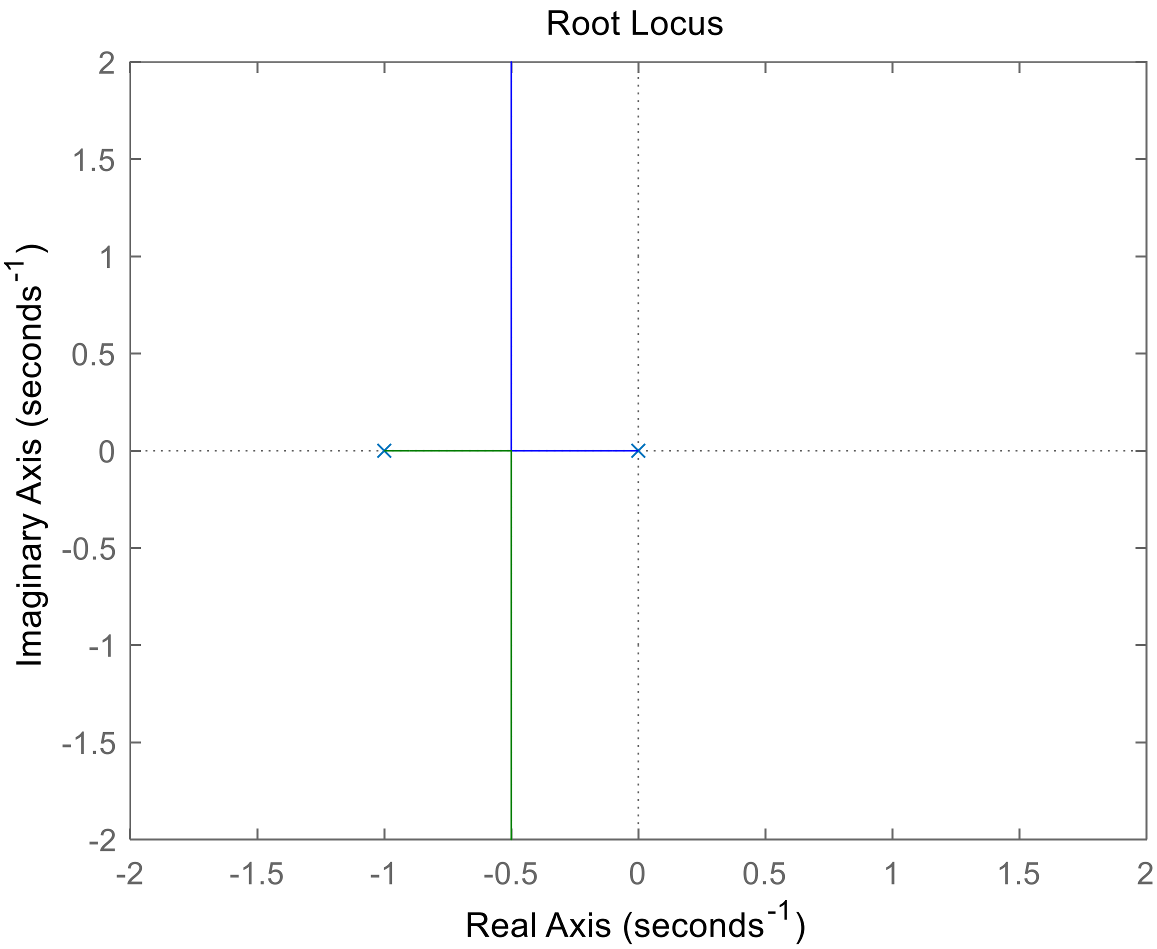


$$1 + KG(z) = 0$$

```
>> T=1;
>> numC=1;denC=[1 1 0];
>> sysC=tf(numC,denC);
>> sysD=c2d(sysC,T)

Transfer function:
  0.3679 z + 0.2642
----------------------
z^2 - 1.368 z + 0.3679

Sampling time: 1
>> rlocus(sysD)
>> zgrid
```


Root Locus
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
0.5π/T
0.6π/T
0.4π/T
0.7π/T
0.3π/T
0.8π/T
0.2π/T
0.9π/T
0.1π/T
π/T
π/T
0.9π/T
0.1π/T
0.8π/T
0.2π/T
0.7π/T
0.3π/T
0.6π/T
0.4π/T
0.5π/T
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

>> rlocus(sysC)



Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

Frequency Response of Digital System

$$Y(z) = G(z)U(z)$$

$$U(z) = \mathcal{Z}\left[\sin \omega t\right] = \frac{z \sin \omega T}{\left(z - e^{j\omega T}\right)\left(z - e^{-j\omega T}\right)}$$

$$Y(z) = \frac{G(z) z \sin \omega T}{\left(z - e^{j\omega T}\right)\left(z - e^{-j\omega T}\right)} = \frac{k_1 z}{z - e^{j\omega T}} + \frac{k_2 z}{z - e^{-j\omega T}} + Y_g(z)$$

$$Y_{ss}(z) = \frac{k_1 z}{z - e^{j\omega T}} + \frac{k_2 z}{z - e^{-j\omega T}}$$

$$k_1 = \frac{G(e^{j\omega T}) \sin \omega T}{e^{j\omega T} - e^{-j\omega T}} = \frac{G(e^{j\omega T})}{2j}$$

Frequency Response of Digital System

$$G(e^{j\omega T}) = \left|G(e^{j\omega T})\right| e^{j\theta}, \theta = \measuredangle G(e^{j\omega T})$$

$$k_1 = \frac{\left|G(e^{j\omega T})\right| e^{j\theta}}{2j}$$

$$k_2 = \frac{\left|G(e^{j\omega T})\right| e^{-j\theta}}{-2j} = -\frac{\left|G(e^{j\omega T})\right| e^{-j\theta}}{2j}$$

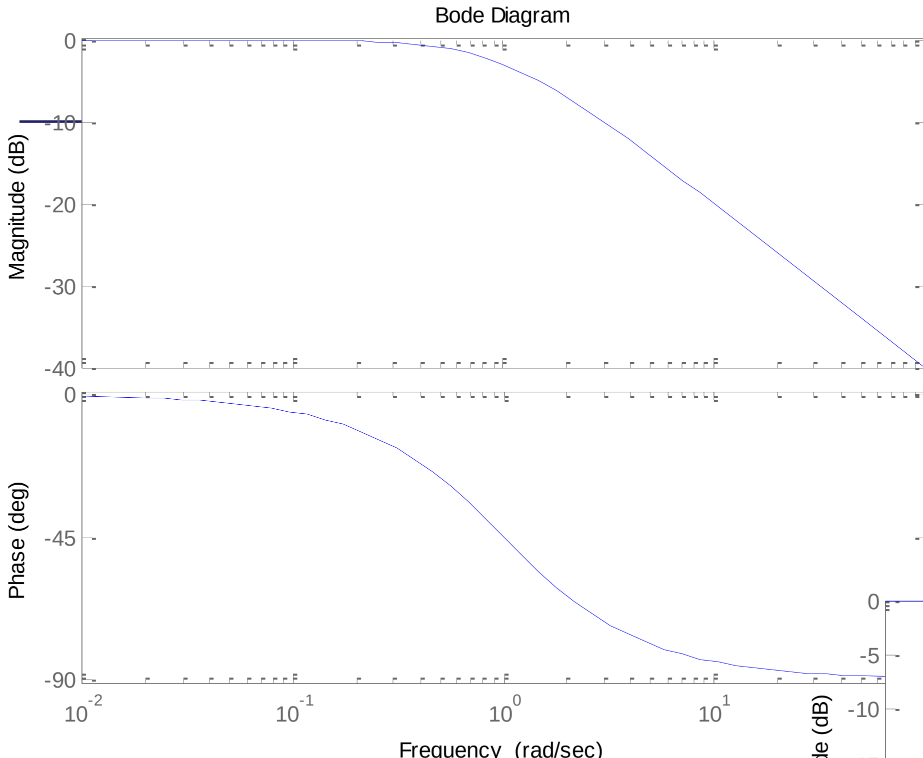
$$y_{ss}(kT) = k_1 (e^{j\omega T})^k + k_2 (e^{-j\omega T})^k = \left|G(e^{j\omega T})\right| \sin(\omega kT + \theta)$$

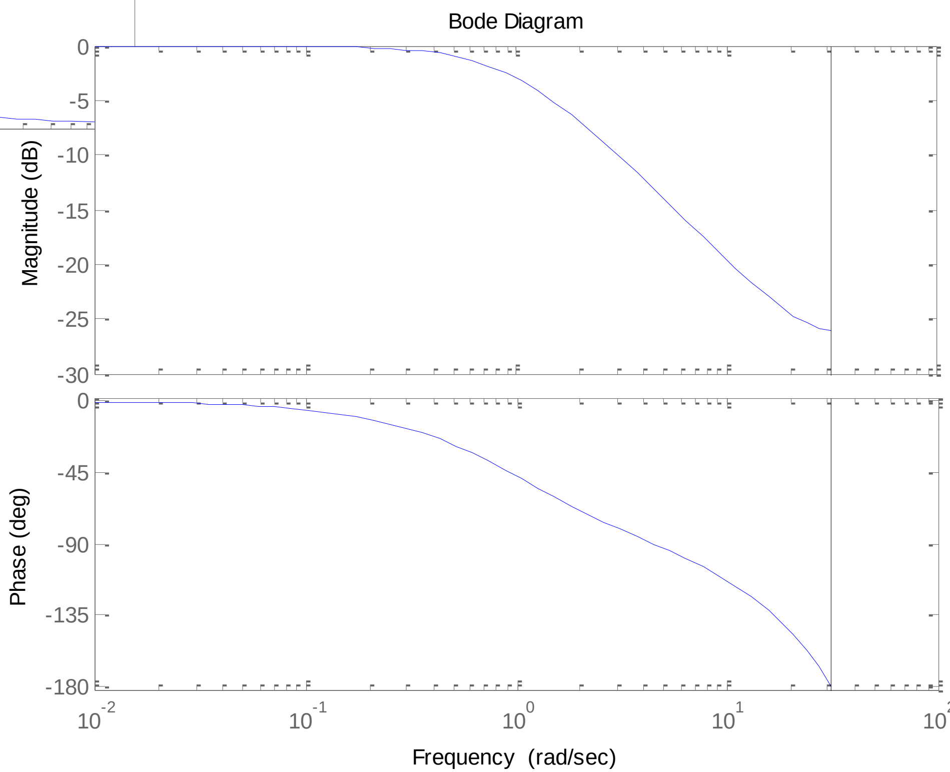
Frequency Response of Digital System

$$G(z) \rightarrow \left| G(e^{j\omega T}) \right|, \measuredangle G(e^{j\omega T})$$

$$G(s) \rightarrow \left| G(j\omega) \right|, \measuredangle G(j\omega)$$

```
>> G=tf(1,[1 1])

Transfer function:
  1
-----
s + 1

>> Gz=c2d(G,0.1)

Transfer function:
 0.09516
----------
z - 0.9048

Sampling time: 0.1
>> figure;bode(G)
>> figure;bode(Gz)
```


Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
Phase (deg)
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)


Bode Diagram
Magnitude (dB)
0
-5
-10
-15
-20
-25
-30
Phase (deg)
0
-45
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Bilinear Transformation

$$z = \frac{1+(T/2)w}{1-(T/2)w}$$

$$w = \frac{2}{T}\frac{z-1}{z+1}$$

On the unit circle in z-plane: $z = e^{j\omega T}$

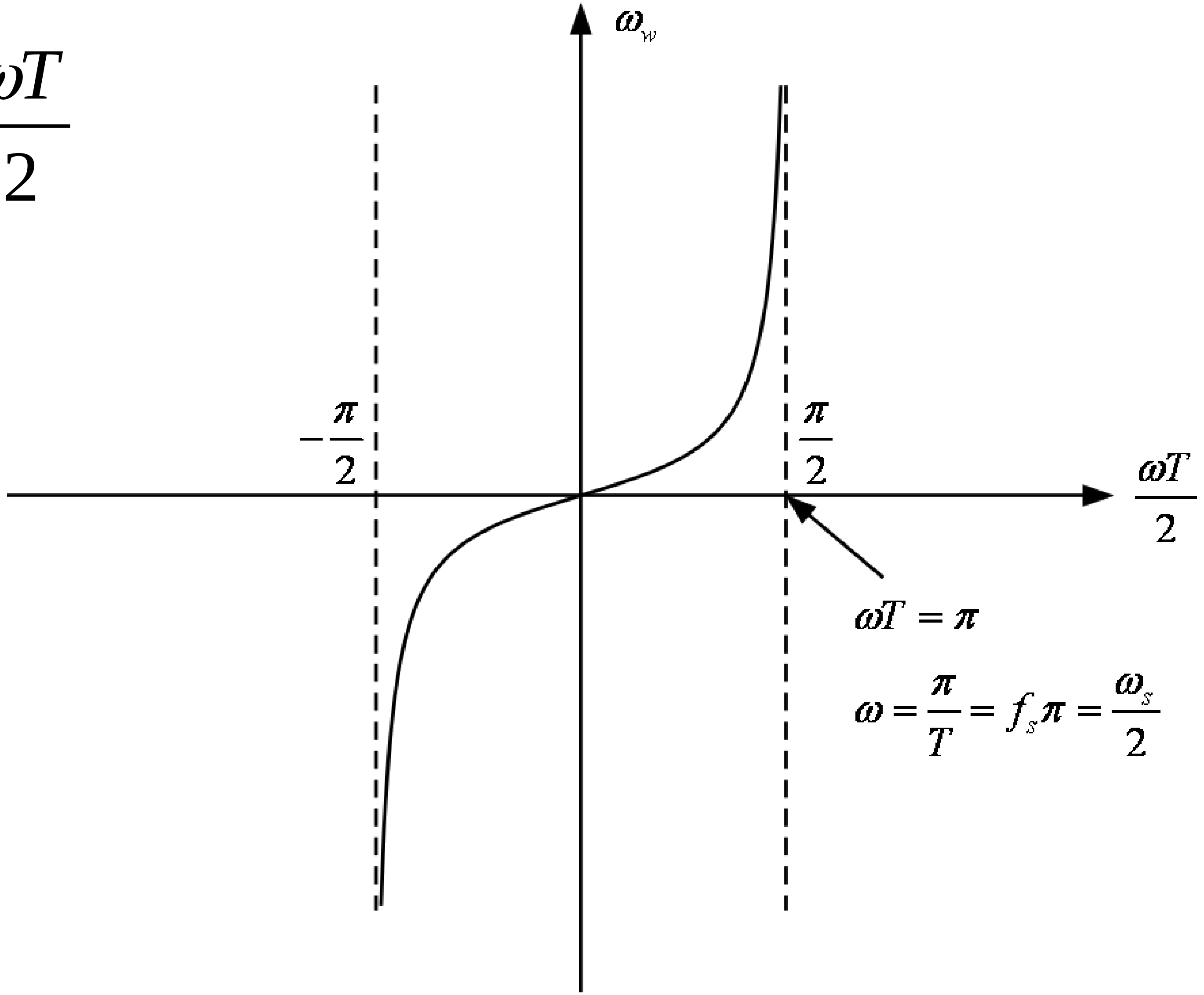
$$w = \frac{2}{T}\frac{z-1}{z+1}\bigg|_{z=e^{j\omega T}} = \frac{2}{T}\frac{e^{j\omega T}-1}{e^{j\omega T}+1} = \frac{2}{T}\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{e^{j\omega T/2}+e^{-j\omega T/2}}$$

$$= j\frac{2}{T}\tan\frac{\omega T}{2} = j\omega_w$$

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2}$$

Bilinear Transformation

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2}$$



Bilinear Transformation

$$\omega << \omega_s$$

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2} = \frac{2}{T}\frac{\sin\frac{\omega T}{2}}{\cos\frac{\omega T}{2}} \approx \frac{2}{T}\frac{\omega T}{2} = \omega$$

$$\frac{\omega T}{2} < \frac{\pi}{10} \rightarrow \text{error is less than 4\%}$$

$$\omega < \frac{2\pi}{10T} = \frac{\omega_s}{10}$$

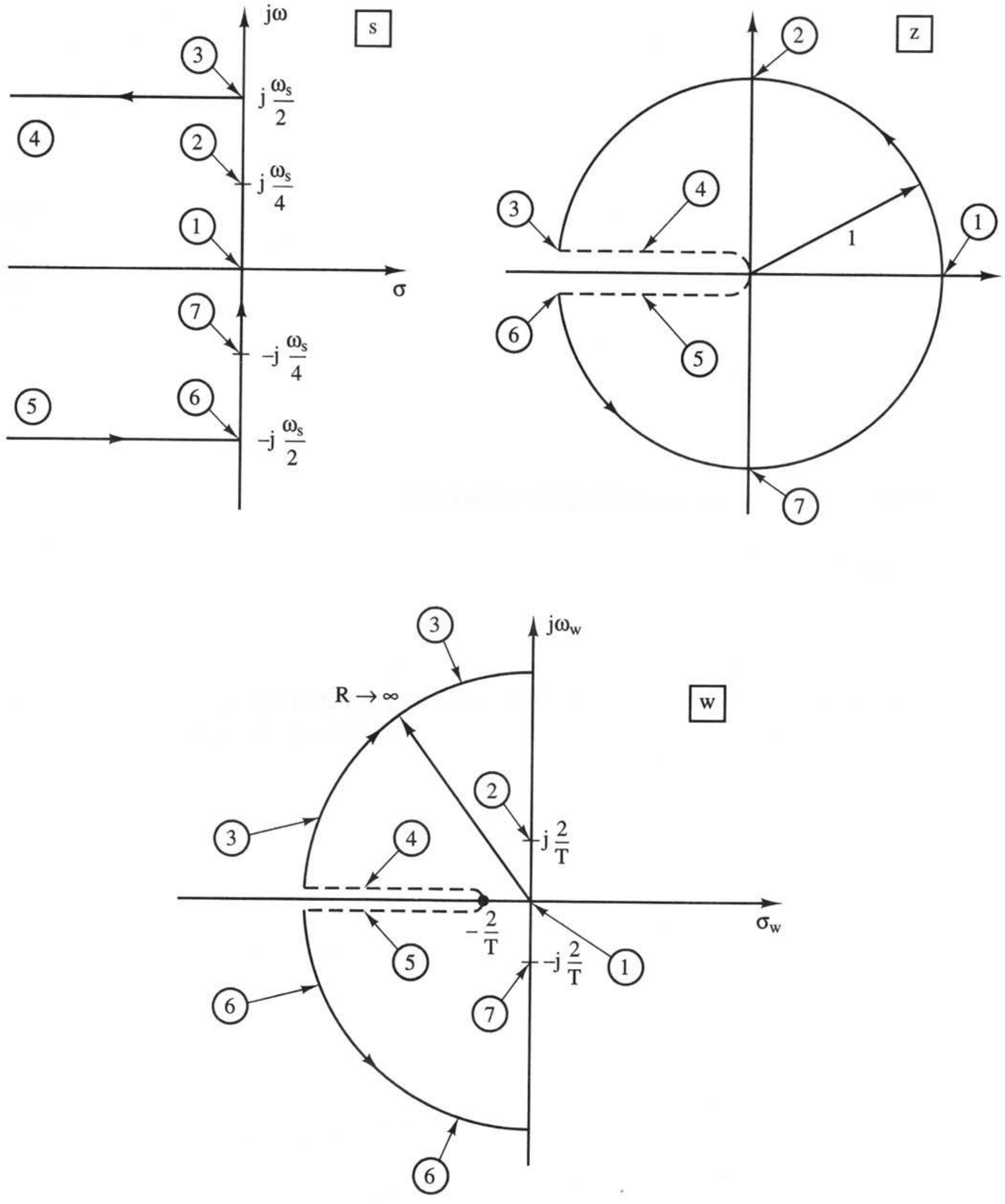


Figure 7-4 Mapping from *s*-plane to *z*-plane to *w*-plane.

MATLAB lead_digital.m

```
N=200;
Ts=1/100
Gp=tf(100,[1 1 0])
Gz=c2d(Gp,Ts,'zoh')

figure(1)
step(feedback(Gz,1),10)

Gw=d2c(Gz,'tustin')
figure(2)
margin(Gw);

w=logspace(-2,3,N);
[mag,phase]=bode(Gw,w);
for i=1:N
   Gw_mag(i)=mag(i);
   Gw_phase(i)=phase(i);
end
```

MATLAB lead_digital.m

```
phimax=50;
alpha=(1-sin(pi*phimax/180))/(1+sin(pi*phimax/180))
10*log10(1/alpha)

[w' (20*log10(Gw_mag))'  Gw_phase']

wmax=16.5;
T=1/(wmax*sqrt(alpha));
num1=[T 1];
den1=[T*alpha 1];

Dw=tf(num1,den1)

figure(3)
margin(Dw*Gw)

Dz=c2d(Dw,Ts,'tustin')
figure(4)
step(feedback(Dz*Gz,1),10)
```

MATLAB lead_digital.m

```
Transfer function:
  100
-------
s^2 + s


Transfer function:
0.004983 z + 0.004967
---------------------
 z^2 - 1.99 z + 0.99

Sampling time: 0.01

Transfer function:
-4.167e-006 s^2 - 0.4992 s + 100
--------------------------------
     s^2 + s + 1.11e-013
```

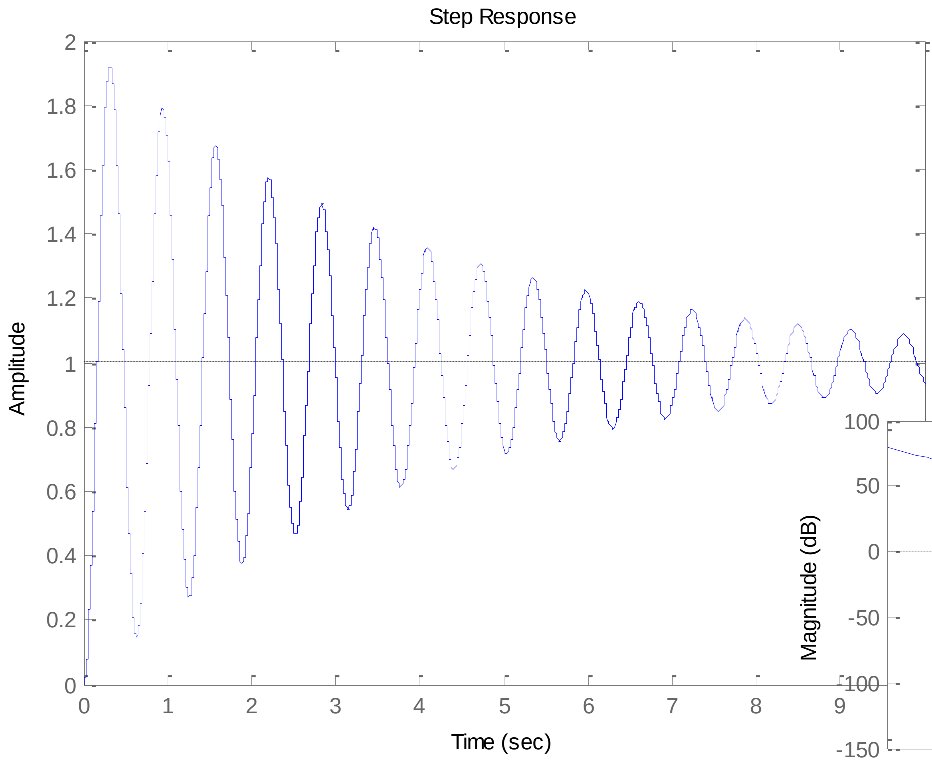
MATLAB lead_digital.m

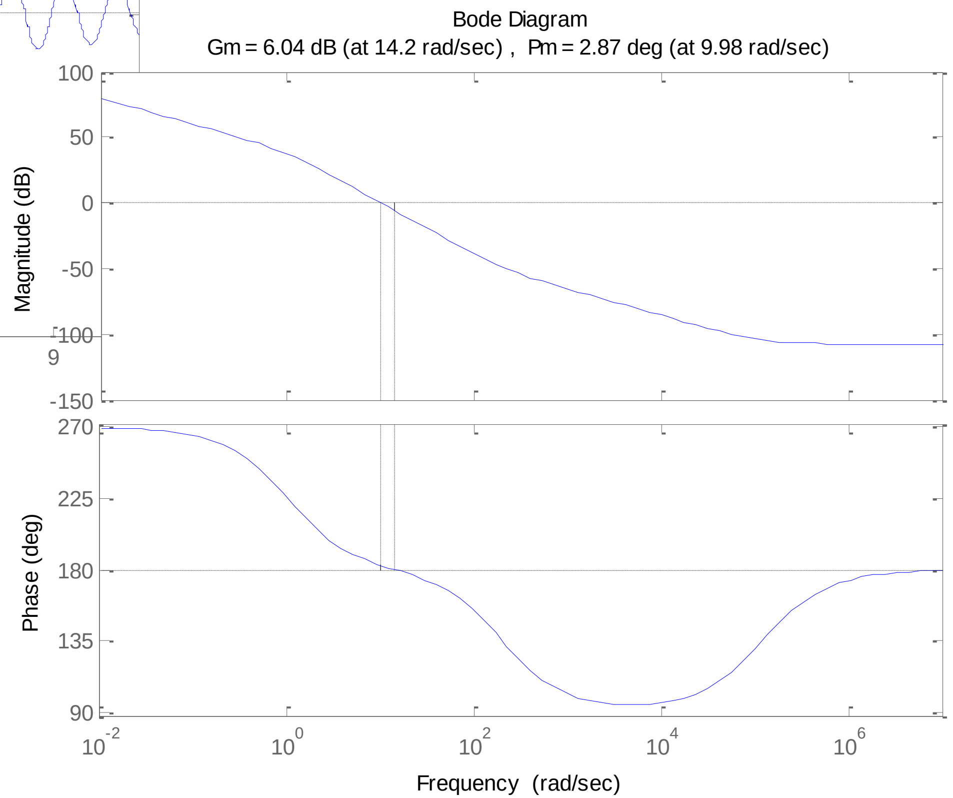
```
alpha =

  1.3247e-001


ans =

  8.7787e+000


ans =

  1.0000e-002  8.0000e+001 -9.0576e+001


  1.4650e+001 -6.6302e+000 -1.8028e+002
  1.5522e+001 -7.6302e+000 -1.8074e+002
  1.6447e+001 -8.6300e+000 -1.8121e+002
  1.7426e+001 -9.6297e+000 -1.8169e+002
```

MATLAB lead_digital.m

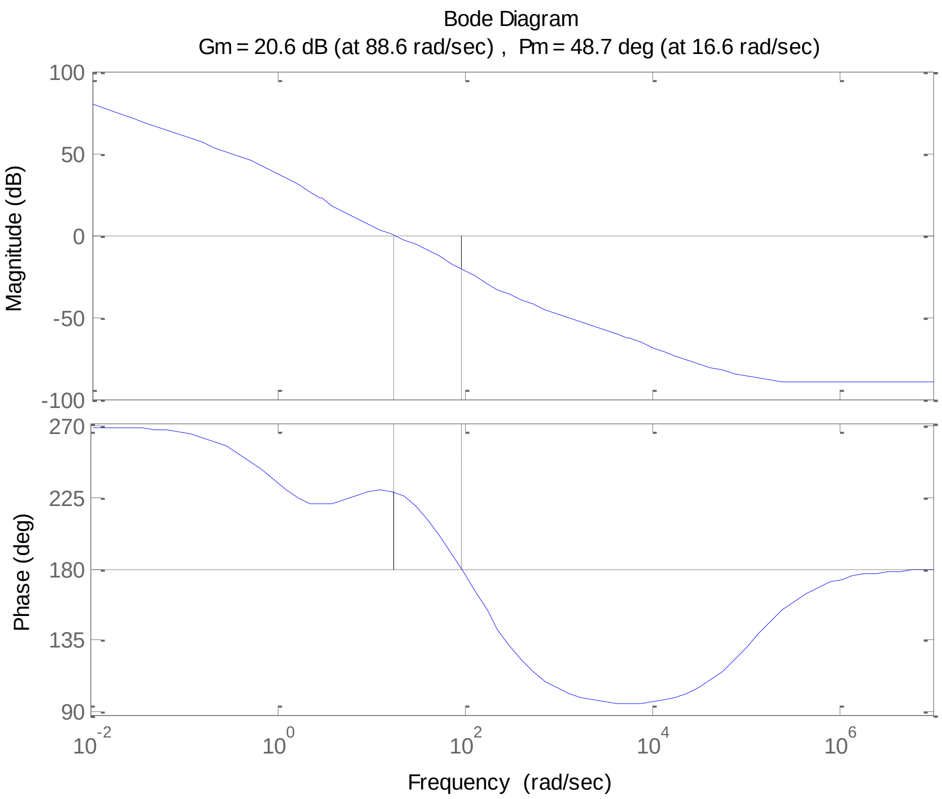
```
Transfer function:
0.1665 s + 1
--------------
0.02206 s + 1


Transfer function:
6.339 z - 5.969
----------------
  z - 0.6304

Sampling time: 0.01
```

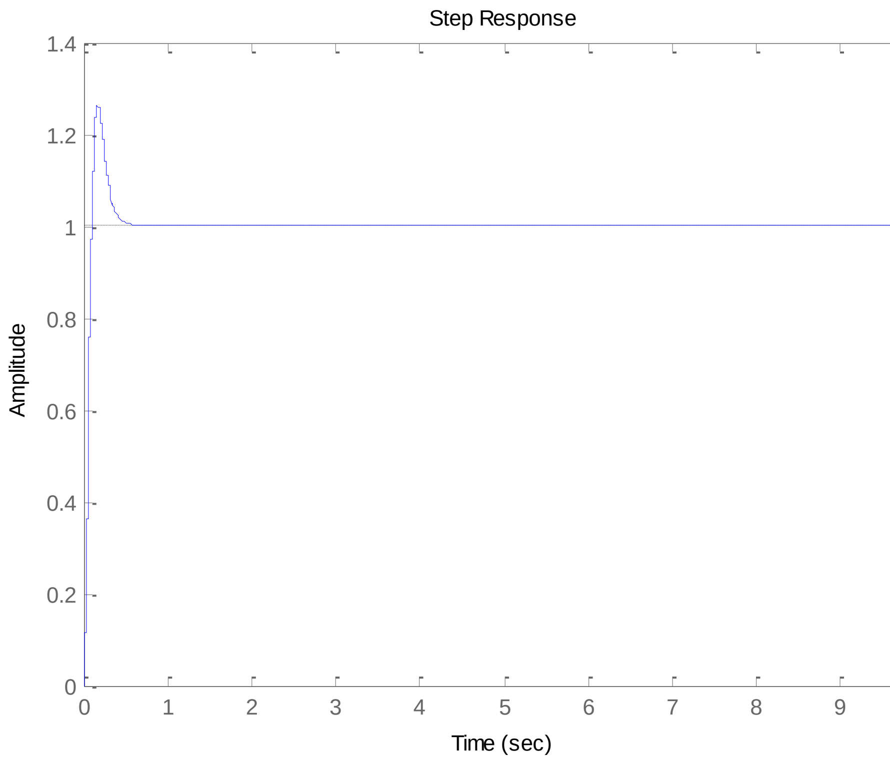
MATLAB lead_digital.m





MATLAB lead_digital.m





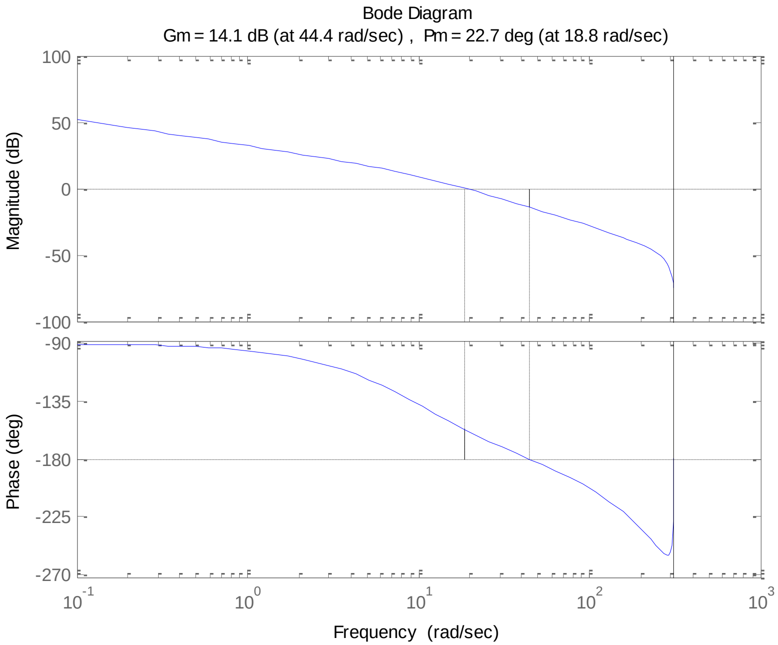
Gz 와 Gw 의 GM 과 PM

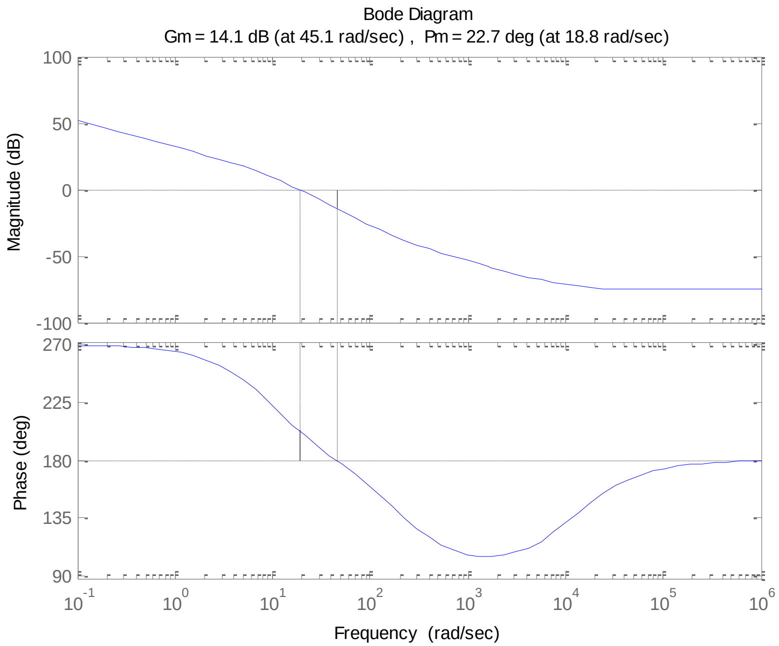
```
N=200;
Ts=1/100
Gp=tf(40,[0.1 1 0])
Gz=c2d(Gp,Ts,'zoh')

figure(1)
margin(Gz);

Gw=d2c(Gz,'tustin')
figure(2)
margin(Gw);

w=logspace(-2,2,N);
[mag,phase]=bode(Gw,w);
for i=1:N
    Gw_mag(i)=mag(i);
    Gw_phase(i)=phase(i);
end
```

Gz 와 Gw 의 GM 과 PM

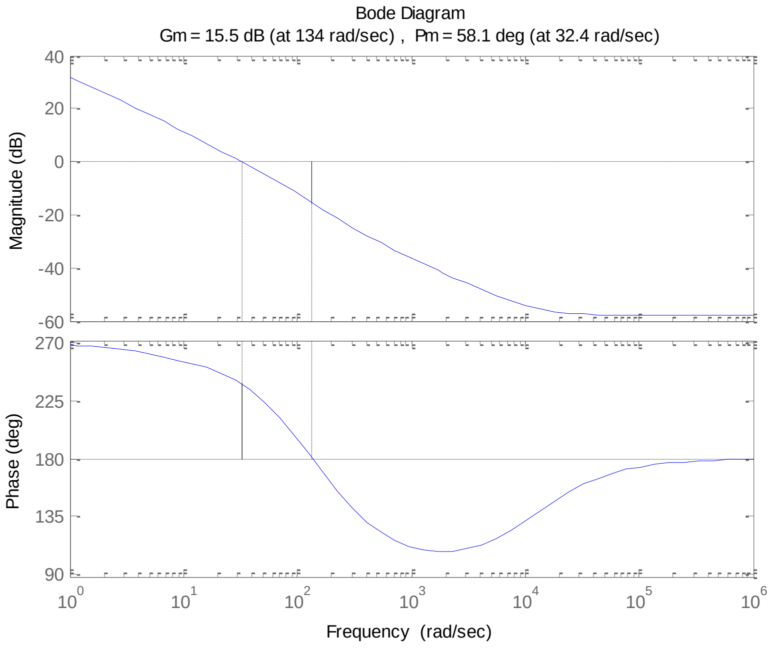
```
phimax=50;
alpha=(1-sin(pi*phimax/180))/(1+sin(pi*phimax/180))
10*log10(1/alpha)

[w' (20*log10(Gw_mag))'  Gw_phase']

wmax=33;
T=1/(wmax*sqrt(alpha));
num1=[T 1];
den1=[T*alpha 1];

Dw=tf(num1,den1)

figure(3)
margin(Dw*Gw)

Dz=c2d(Dw,Ts,'tustin')
figure(4)
margin(Dz*Gz)
```

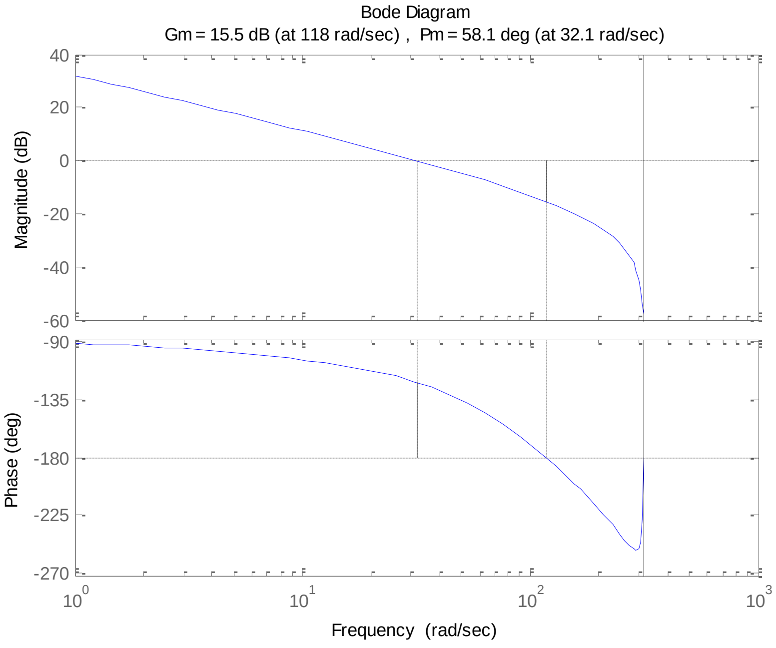
Gz 와 Gw 의 GM 과 PM





Gz 와 Gw 의 GM 과 PM





$$z = \frac{1 + (T/2)w}{1 - (T/2)w}$$

$$w = \frac{2}{T}\frac{z-1}{z+1}$$

On the unit circle in z-plane: $z = e^{j\omega T}$

$$w = \frac{2}{T}\frac{z-1}{z+1}\bigg|_{z=e^{j\omega T}} = \frac{2}{T}\frac{e^{j\omega T}-1}{e^{j\omega T}+1} = \frac{2}{T}\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{e^{j\omega T/2}+e^{-j\omega T/2}}$$

$$= j\frac{2}{T}\tan\frac{\omega T}{2} = j\omega_w$$

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2}$$

$$e^{j\omega T} = \frac{1 + (T/2)j\omega_w}{1 - (T/2)j\omega_w}$$

$$G_w(w) = G(z)\Big|_{z=\frac{1+(T/2)w}{1-(T/2)w}} = G\left(\frac{1+(T/2)w}{1-(T/2)w}\right)$$

$$G_w(j\omega_w) = G\left(\frac{1+(T/2)j\omega_w}{1-(T/2)j\omega_w}\right) = G(e^{j\omega T})$$