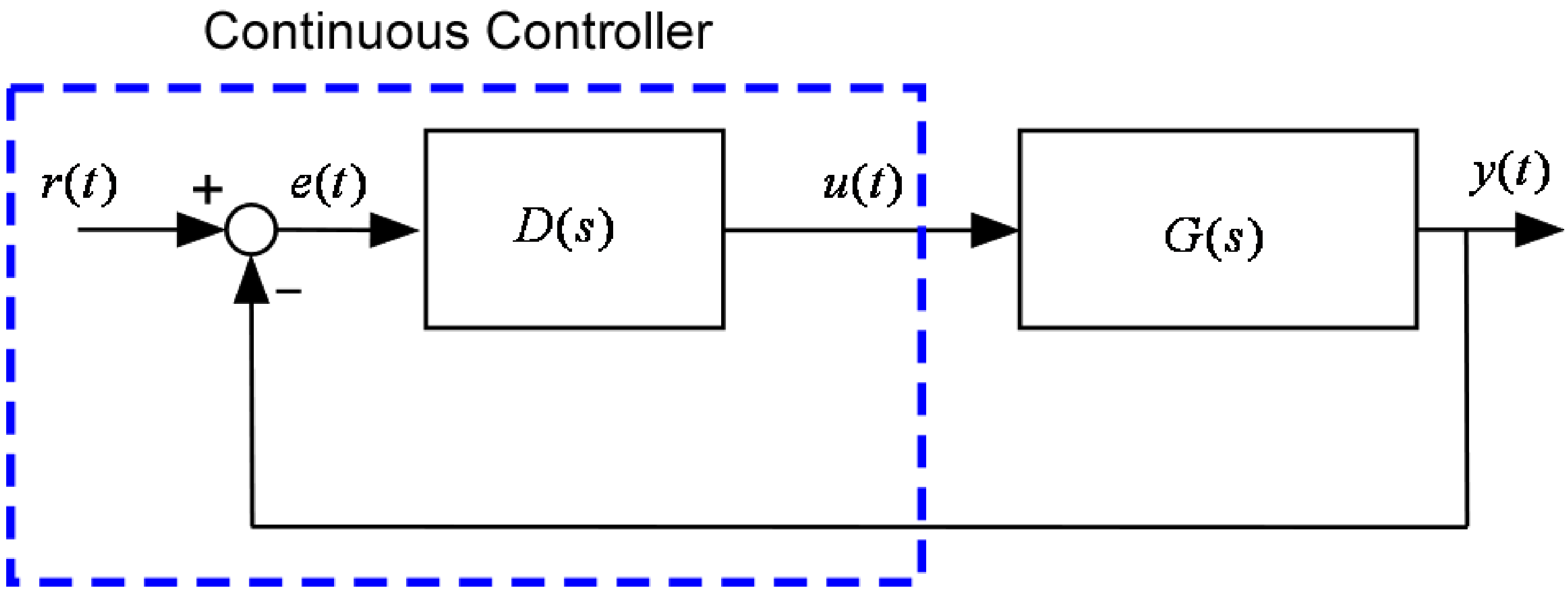
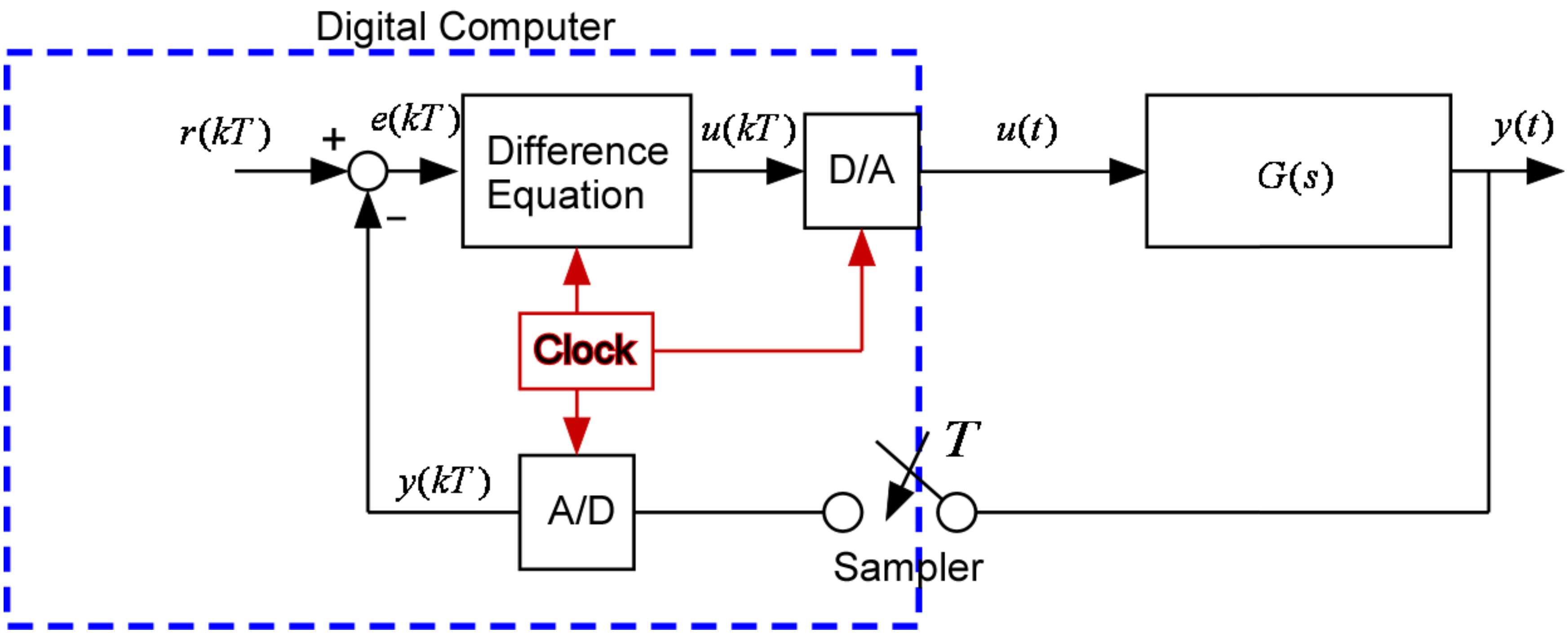
Digitization & Digital Control

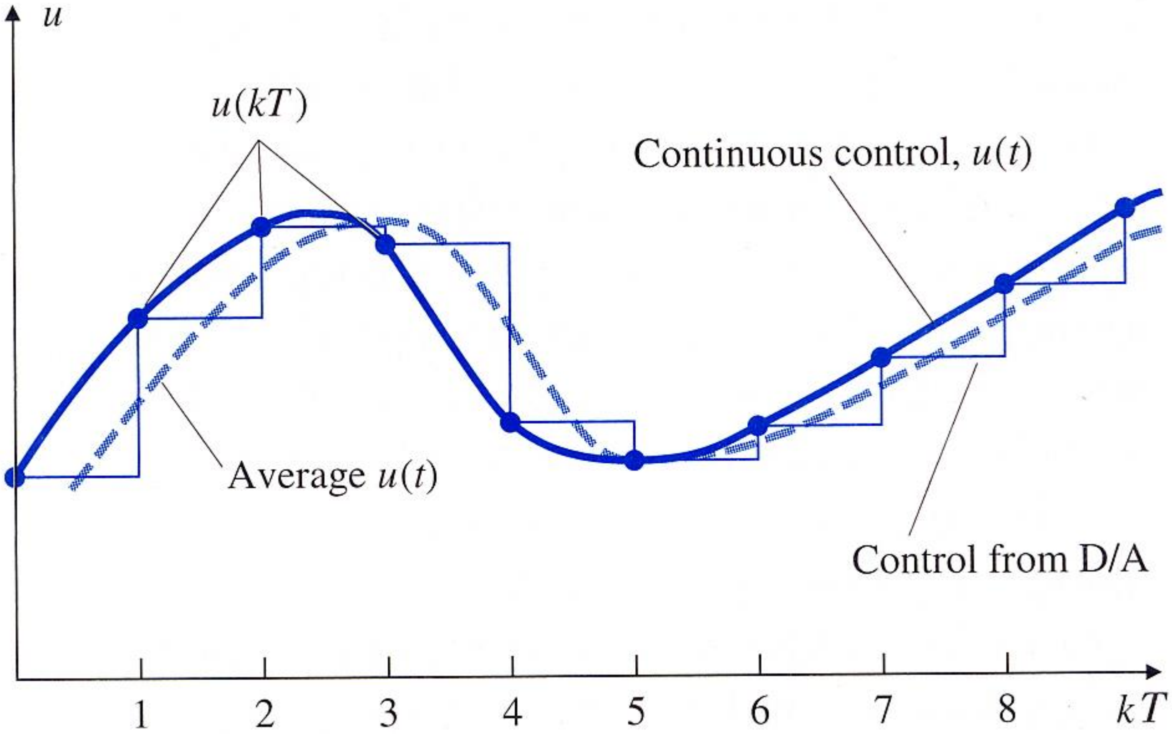
Digitization



Digitization



Digitization



- ZOH(zero order hold)

Laplace Transforms

- Laplace Transform

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$\mathcal{L}[\dot{f}(t)] = sF(s)$$

$$\ddot{y}(t) + a\dot{y}(t) + by(t) = u(t)$$

$$s^2Y(s) + asY(s) + bY(s) = U(s)$$

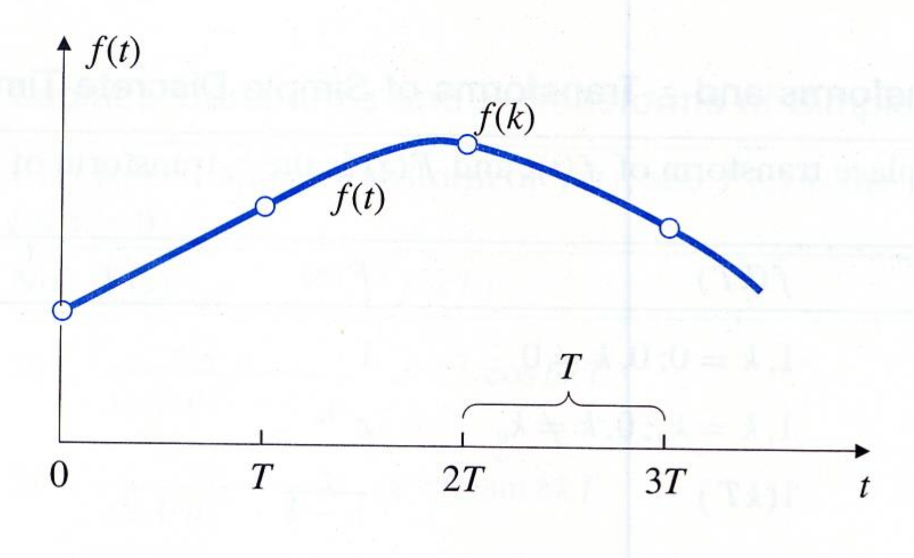
$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + as + b}$$

z-Transform

- z-Transform

$$\mathcal{Z}[f(k)] = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$\mathcal{Z}[f(k-1)] = z^{-1} F(z)$$



z-Transform

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z)$$

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

z-transform example

$$e(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$E(z) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \left| z^{-1} \right| < 1$$

$$e(k) = \begin{cases} a^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$E(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \left| az^{-1} \right| < 1$$

Inverse z-tranform

$$E(z) = \frac{z}{(z-1)(z-2)}$$

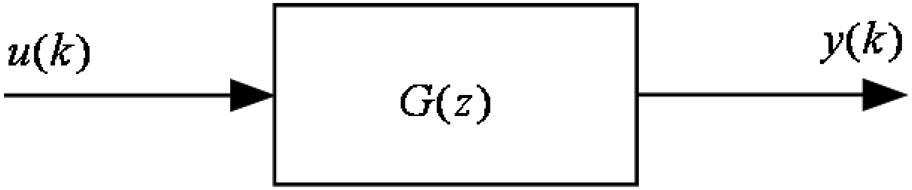
$$\frac{E(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$E(z) = \frac{-z}{z-1} + \frac{z}{z-2}$$

$$e(k) = -1 + 2^k, k \geq 0$$

System Response

$$y(k) = ay(k-1) + u(k)$$



$$Y(z) = az^{-1}Y(z) + U(z)$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$U(z) = \frac{z}{z-1}$$

$$Y(z) = G(z)U(z) = \frac{z}{z-a}\frac{z}{z-1}$$

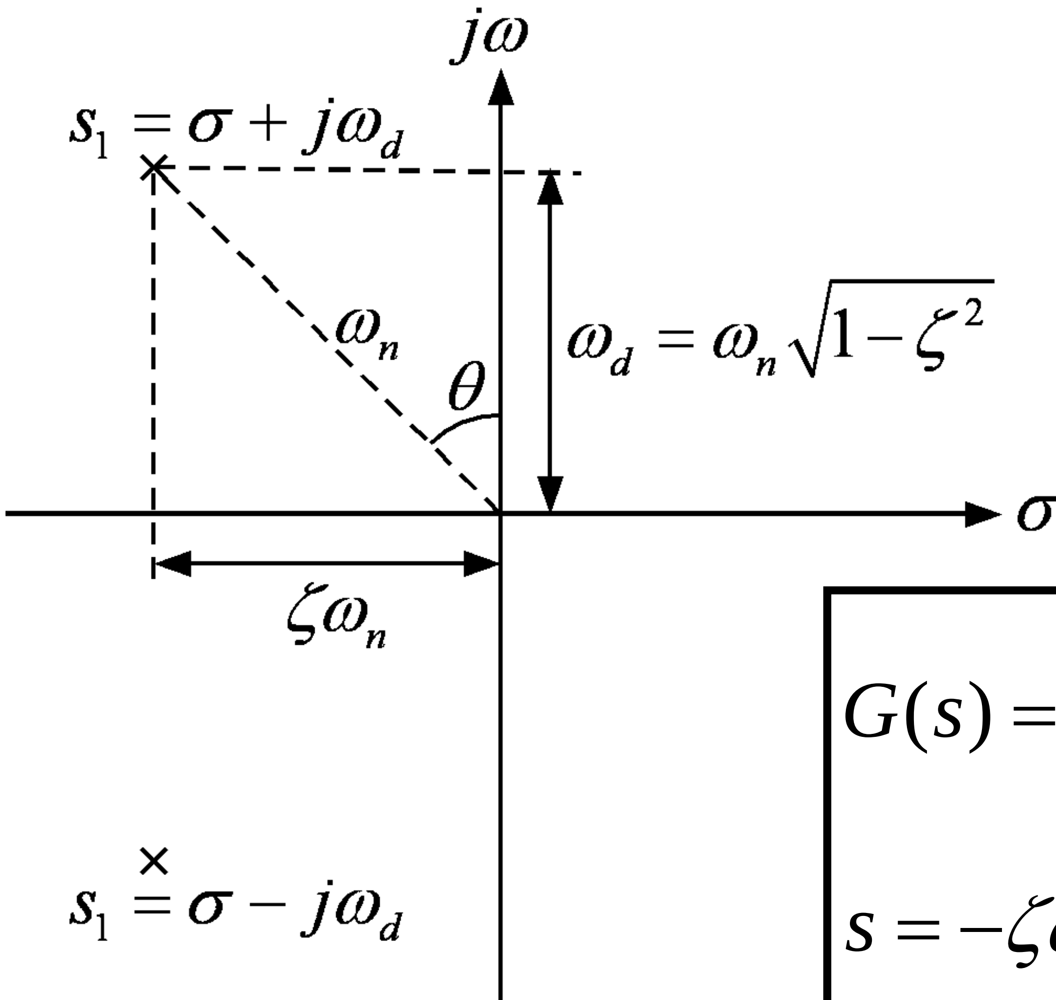
System Response

$$\frac{Y(z)}{z} = \frac{z}{(z-a)(z-1)} = \frac{a/(a-1)}{z-a} + \frac{1/(1-a)}{z-1}$$

$$Y(z) = \frac{za/(a-1)}{z-a} + \frac{z/(1-a)}{z-1}$$

$$y(k) = \frac{1}{1-a} + \frac{a}{a-1} a^k, k \geq 0$$

Continuous-Time System

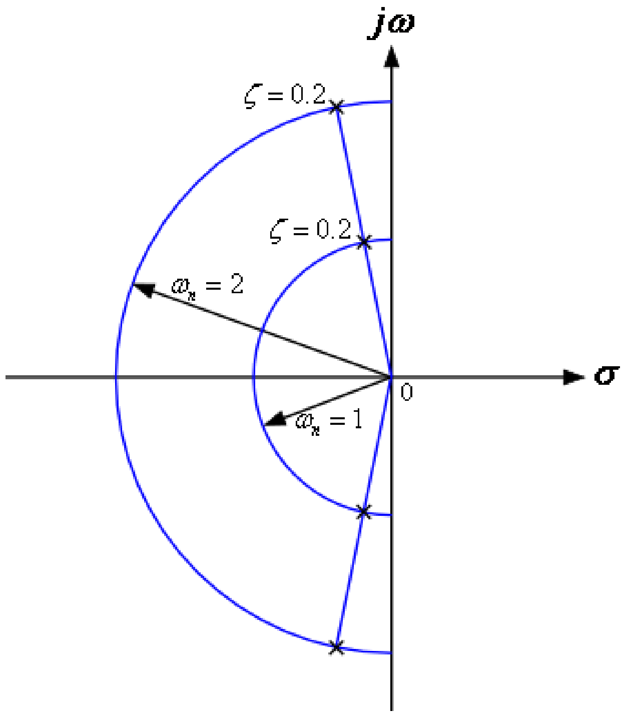


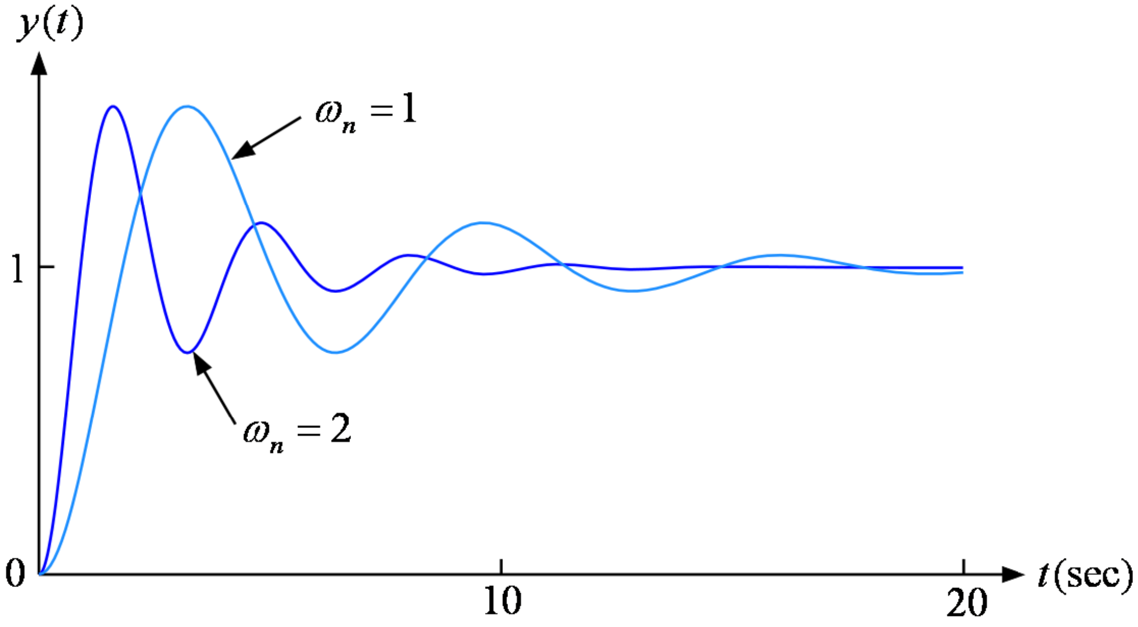
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

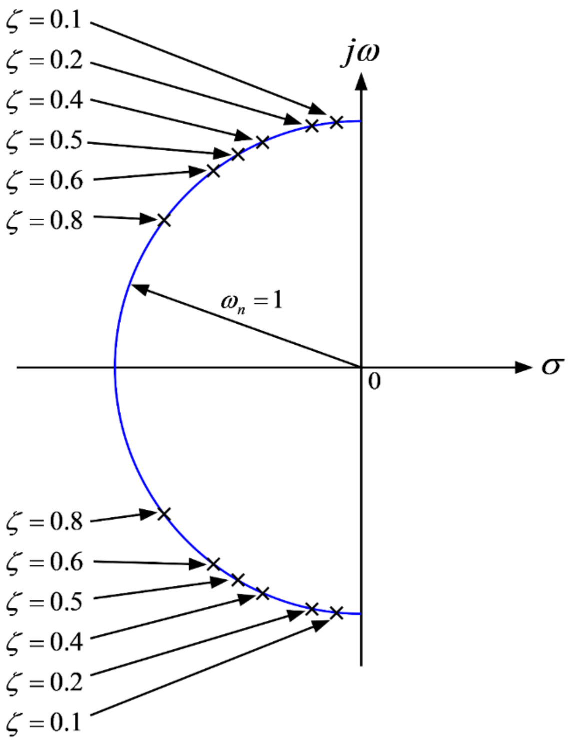
$$= \sigma \pm j\omega_d$$

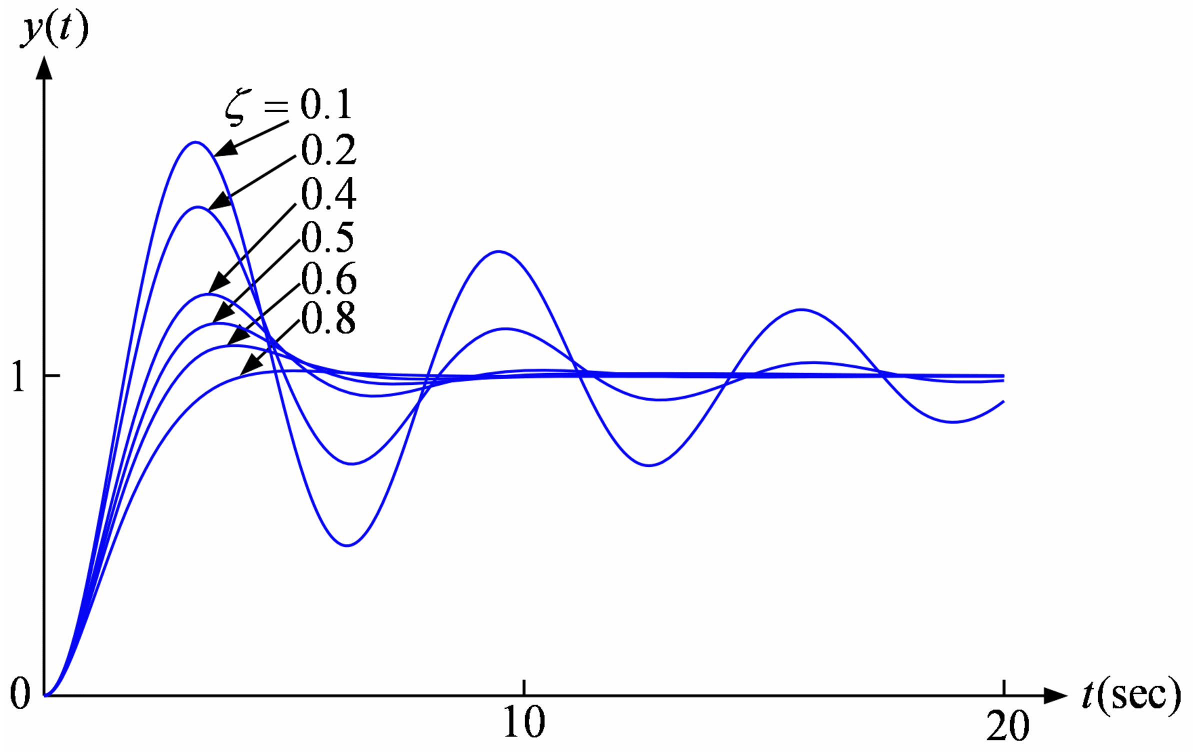
2nd order System





2nd order System





Relationship Between *s* and *z*

$$f(t) = e^{-at}, t \geq 0$$

$$F(s) = \frac{1}{s+a} \rightarrow pole: s = -a$$

$$f(kT) = e^{-akT}, k \geq 0$$

$$\mathcal{Z}[f(k)] = 1 + e^{-aT}z^{-1} + e^{-a2T}z^{-2} + \cdots$$

$$= \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}} \rightarrow pole: z = e^{-aT}$$

$$z = e^{sT}$$

Relationship Between *s* and *z*

$$s = j\omega$$

$$z = e^{j\omega T} = \cos(\omega T) + j\sin(\omega T) = 1\measuredangle(\omega T)$$

$$\omega = \frac{\omega_s}{2} = \frac{2\pi f_s}{2} = \frac{\pi}{T} \rightarrow \omega T = \pi$$

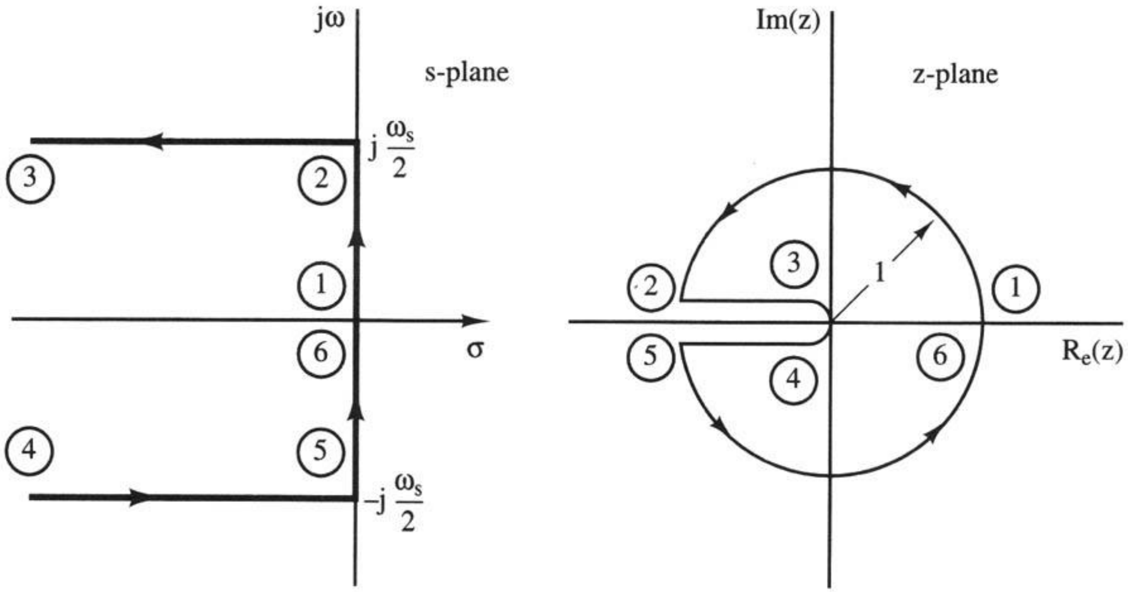


Figure 6-6 Mapping the primary strip into the *z*-plane.

Relationship Between *s* and *z*

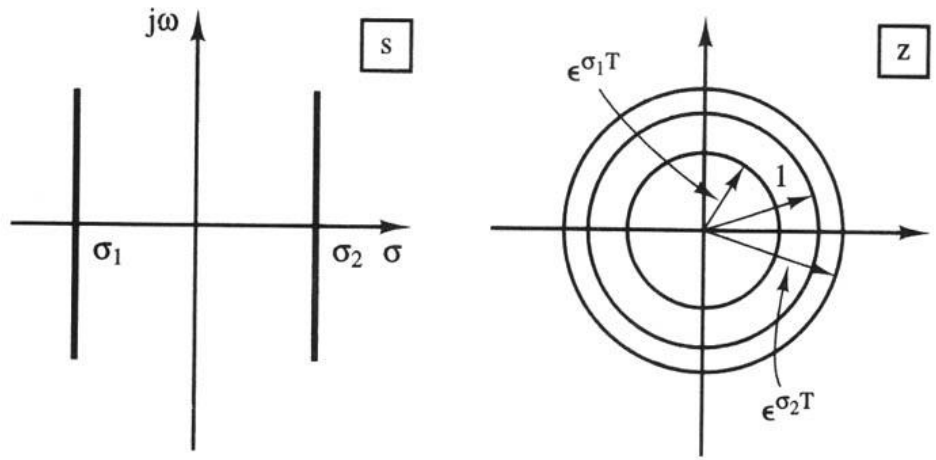


Figure 6-7 Mapping constant damping loci into the *z*-plane.

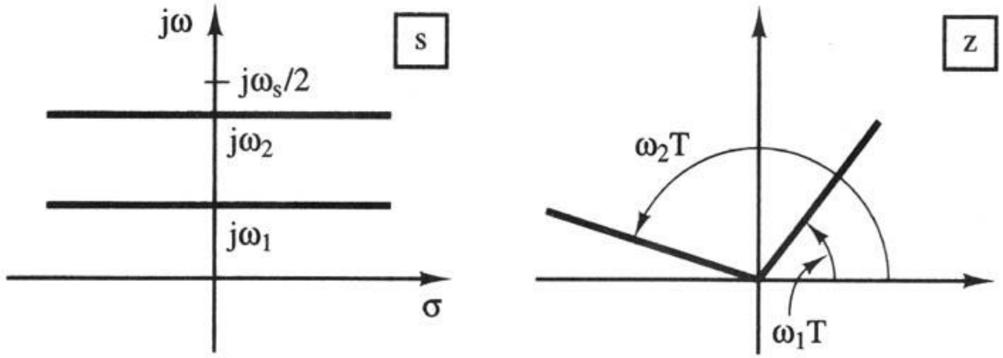


Figure 6-8 Mapping constant frequency loci into the *z*-plane.

Relationship Between *s* and *z*

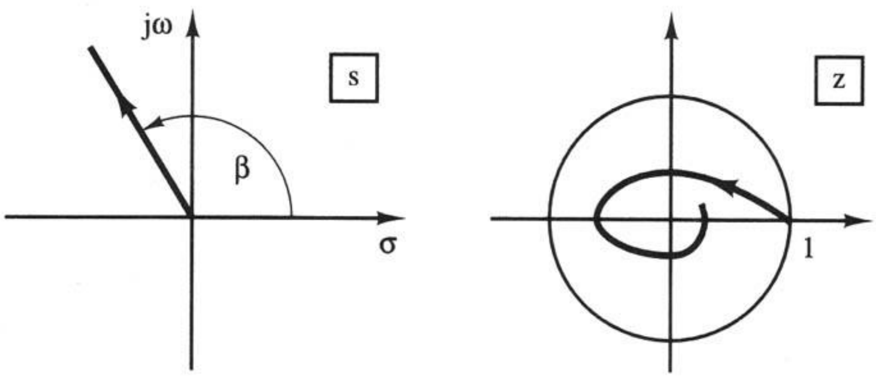
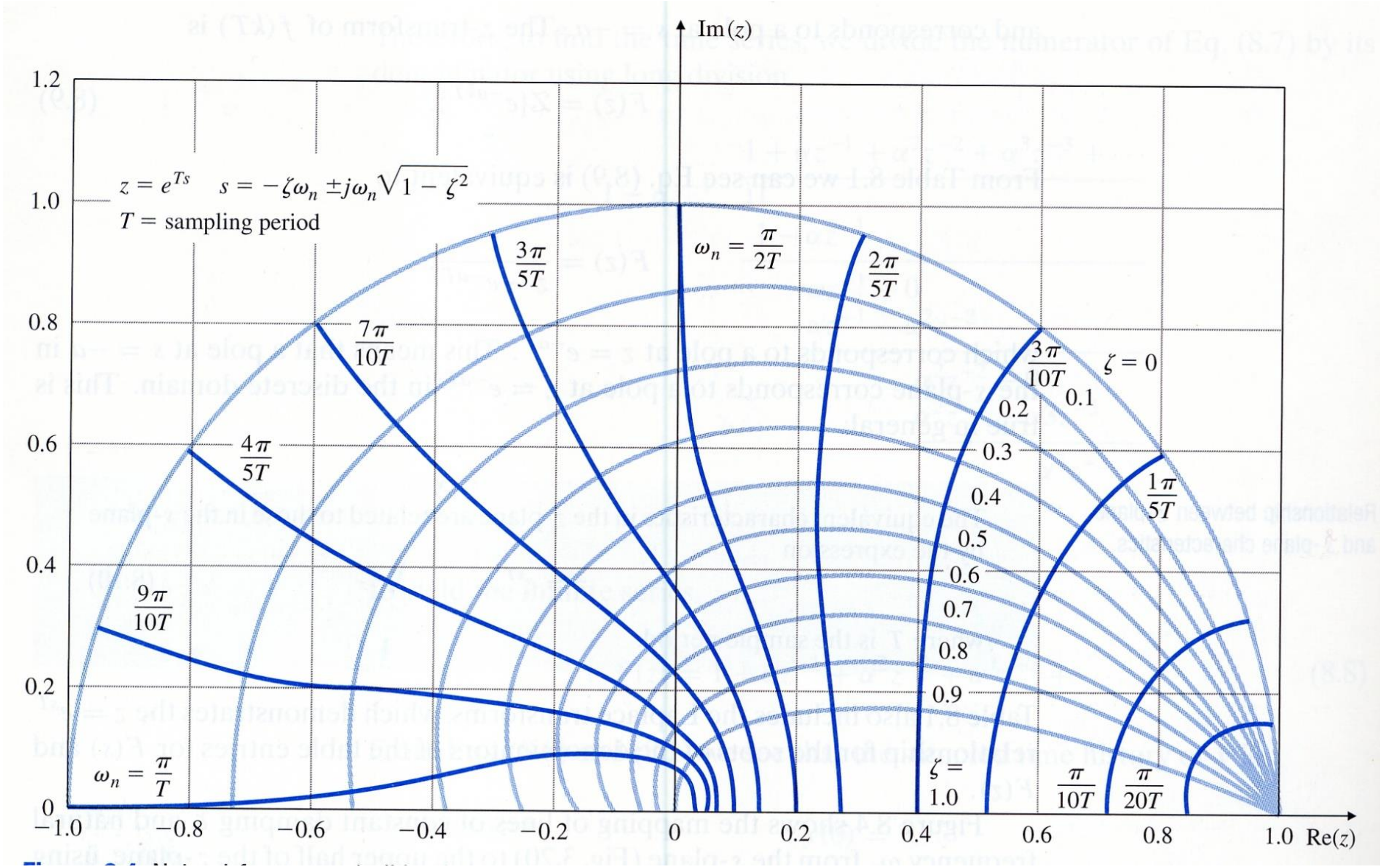


Figure 6-9 Mapping constant damping-ratio loci into the *z*-plane.

Relationship Between *s* and *z*



Mapping s-plane into z-plane

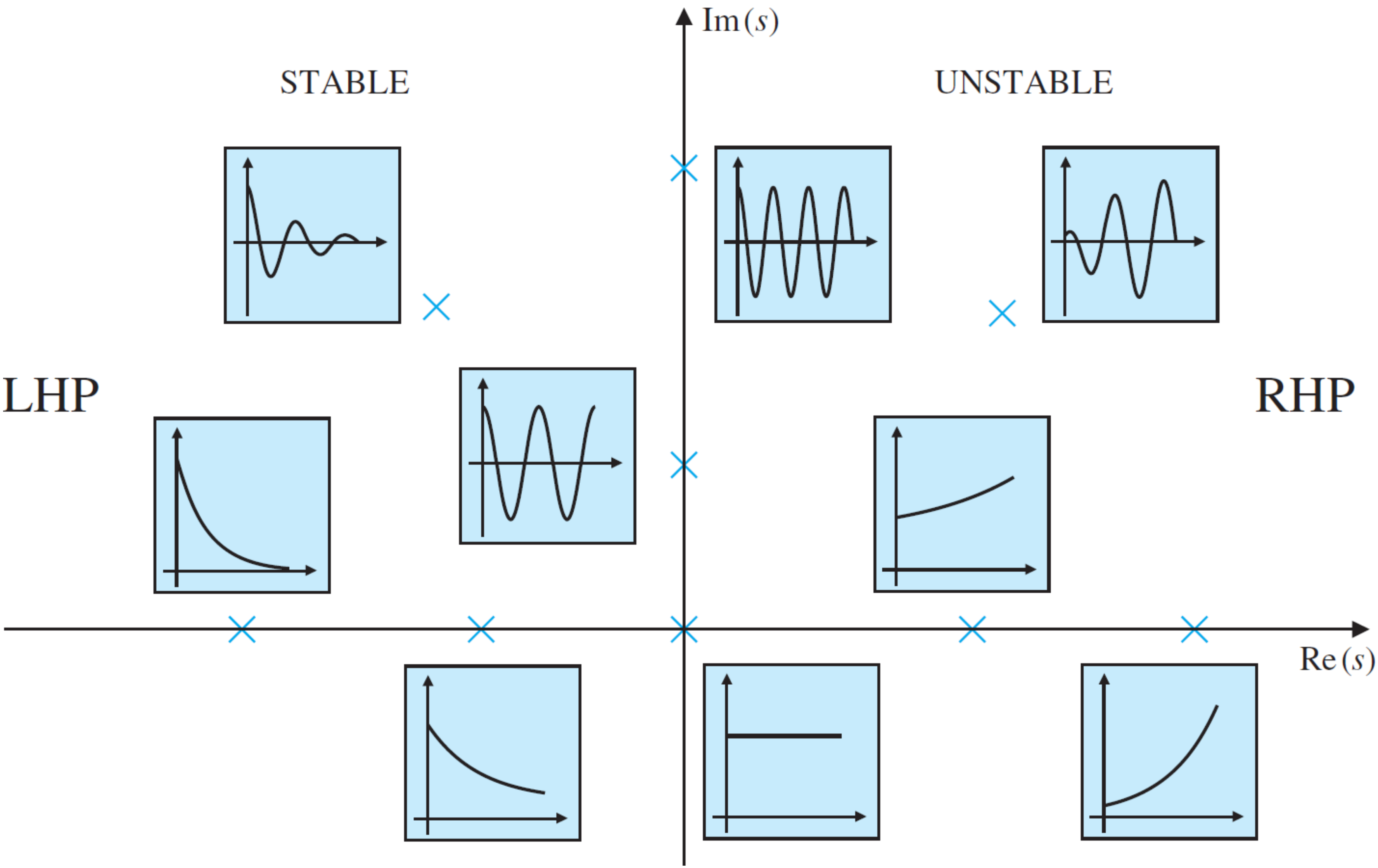
$$z = e^{sT}\Big|_{s=s_{1,2}} = e^{-\zeta\omega_n T} e^{\pm j\omega_n T\sqrt{1-\zeta^2}} = e^{-\zeta\omega_n T} \measuredangle\left(\pm\omega_n T\sqrt{1-\zeta^2}\right) = r\measuredangle(\pm\theta)$$

$$\left.\begin{aligned} e^{-\zeta\omega_n T} = r \rightarrow \zeta\omega_n T = -\ln r \\ \omega_n T\sqrt{1-\zeta^2} = \theta \end{aligned}\right\} \rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{-\ln r}{\theta}$$

$$\zeta = \frac{-\ln r}{\sqrt{\ln^2 r + \theta^2}}, \omega_n = \frac{1}{T}\sqrt{\ln^2 r + \theta^2}$$

$$\tau = \frac{1}{\zeta\omega_n} = \frac{T}{\ln r}$$

Continuous-Time System



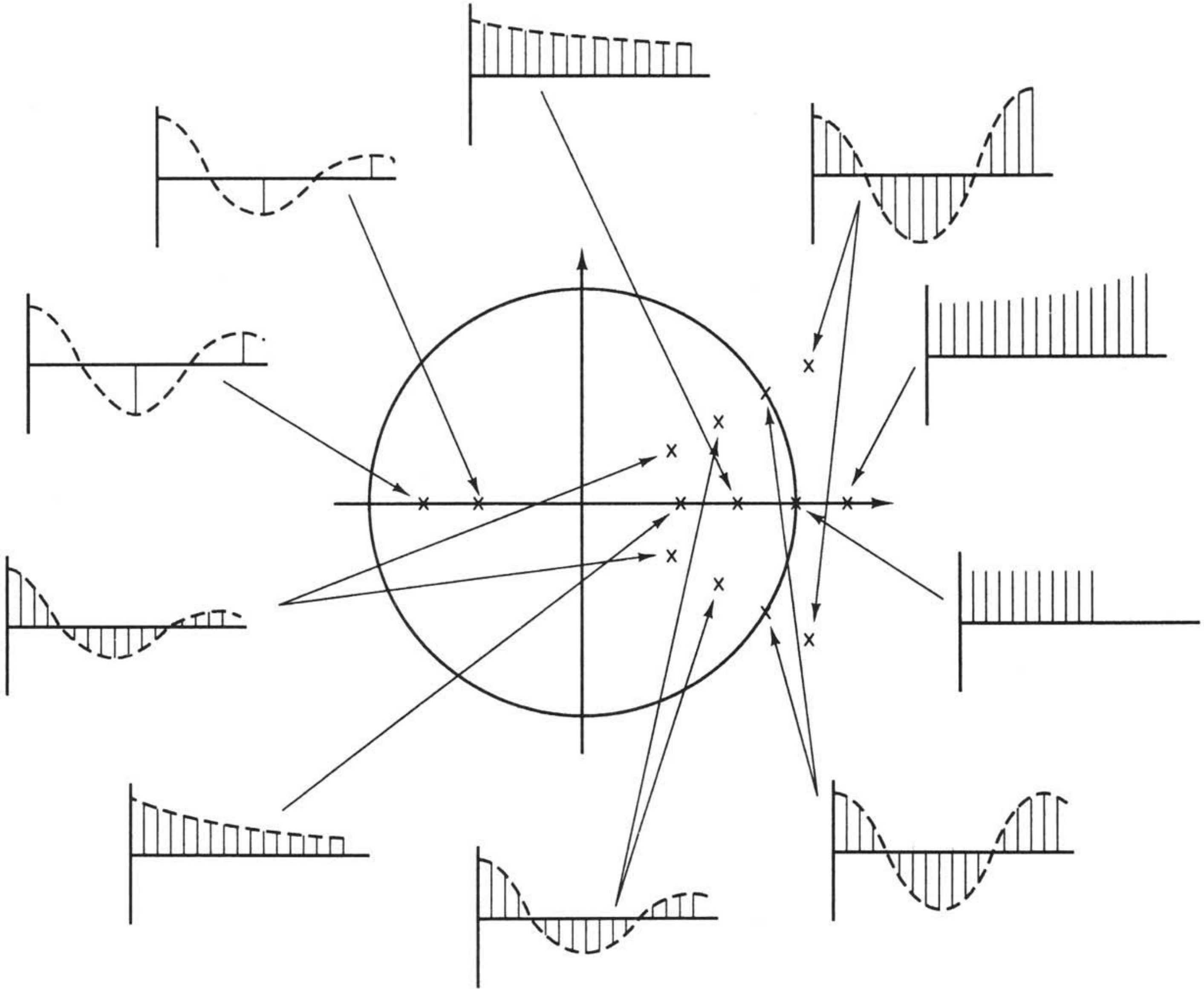


Figure 6-11 Transient response characteristics of the z-plane pole locations.

Final Value Theorem

$$\lim_{t\to\infty} x(t) = x_{ss} = \lim_{s\to 0} sX(s)$$

if the poles of $sX(s)$ are inside the left half plane

$$\lim_{k\to\infty} x(k) = x_{ss} = \lim_{z\to 1}(1-z^{-1})X(z)$$

if the poles of $(1-z^{-1})X(z)$ are inside the unit circle

Final Value Theorem Example: DC gain

$$G(z) = \frac{X(z)}{U(z)} = \frac{0.58(1+z)}{z+0.16}$$

$$t \to \infty$$

$$u(k) = 1, k \geq 0$$

$$U(z) = \frac{1}{1-z^{-1}}$$

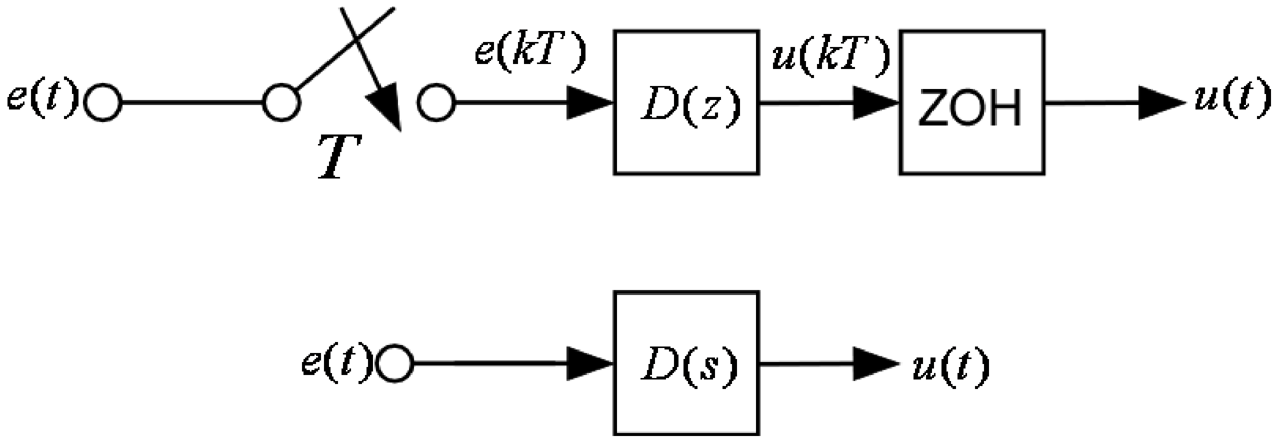
$$X(z) = \frac{0.58(1+z)}{z+0.16}\frac{1}{1-z^{-1}}$$

$$\lim_{k\to\infty} x(k) = x_{ss} = \lim_{z\to 1}(1-z^{-1})X(z) = \lim_{z\to 1}\frac{0.58(1+z)}{z+0.16} = 1$$

Digital Controller Design

- Design by Emulation-Approximation
  - Euler’s Method
  - Tustin’s Method
- Discrete Design (Direct Digital Design)
  - Root Locus
  - Bode Plot

Design by Emulation

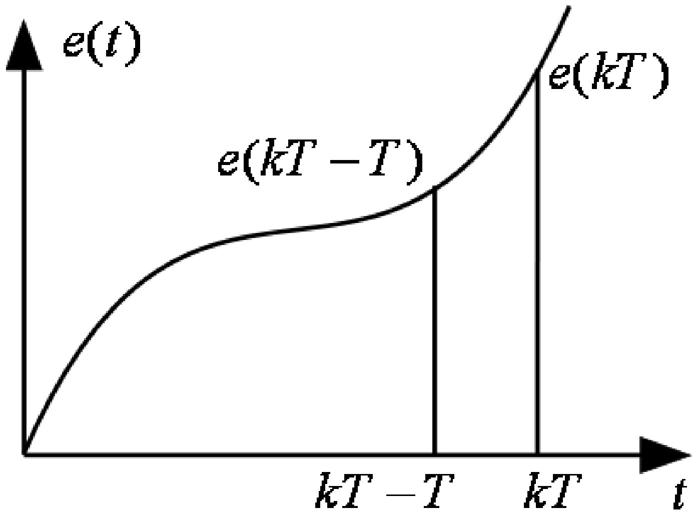


Design by Emulation

$$\frac{U(s)}{E(s)} = D(s) = \frac{1}{s}$$

$$u(kT) = \int_0^{kT} e(t)dt = \int_0^{kT-T} e(t)dt + \int_{kT-T}^{kT} e(t)dt$$

$$= u(kT-T) + \int_{kT-T}^{kT} e(t)dt$$

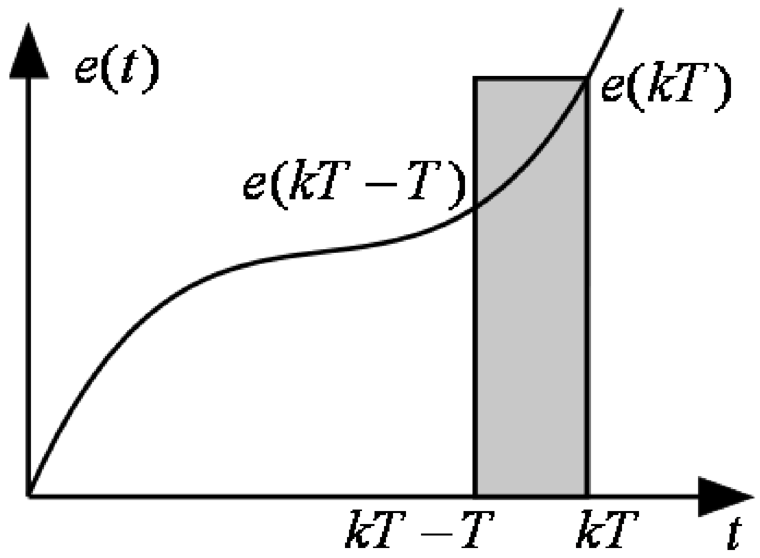


Backward Rectangular Rule (Euler's Method)

$$u(k) = u(k-1) + T \cdot e(k)$$

$$U(z) = z^{-1}U(z) + T \cdot E(z)$$

$$\frac{U(z)}{E(z)} = \frac{T}{1-z^{-1}} = \frac{1}{\left(\frac{1-z^{-1}}{T}\right)}$$



$$D(s) = \frac{a}{s+a}, D(z) = \frac{a}{\frac{1-z^{-1}}{T} + a}$$

$$s \leftarrow \frac{1-z^{-1}}{T}$$

Backward Rectangular Rule(Euler's Method)

$$E(s) = sU(s)$$

$$e(t) = \dot{u}(t)$$

$$e(t) \approx \frac{u(t) - u(t-T)}{T}$$

$$u(t) \approx u(t-T) + T \cdot e(t)$$

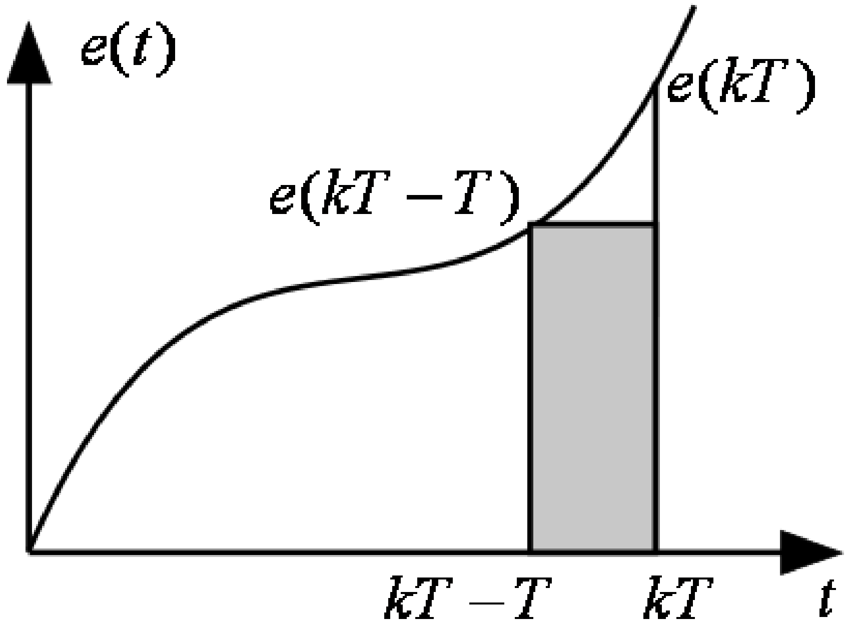
$$u(kT) \approx u(kT-T) + T \cdot e(kT)$$

Forward Rectangular Rule (Euler's Method)

$$u(k) = u(k-1) + T \cdot e(k-1)$$

$$U(z) = z^{-1}U(z) + T \cdot z^{-1}E(z)$$

$$\frac{U(z)}{E(z)} = \frac{Tz^{-1}}{1-z^{-1}} = \frac{T}{z-1} = \frac{1}{\left(\frac{z-1}{T}\right)}$$



$$D(s) = \frac{a}{s+a}, D(z) = \frac{a}{\frac{z-1}{T}+a}$$

$$s \leftarrow \frac{z-1}{T}$$

Forward Rectangular Rule (Euler's Method)

$$E(s) = sU(s)$$

$$e(t) = \dot{u}(t)$$

$$e(t) \approx \frac{u(t+T) - u(t)}{T}$$

$$u(t+T) \approx u(t) + T \cdot e(t)$$

$$u(k+1) \approx u(k) + T \cdot e(k)$$

$$zU(z) = U(z) + TE(z)$$

$$\frac{U(z)}{E(z)} = \frac{T}{z-1}$$

Tustin's Method (Trapezoidal Rule, Bilinear Transform)

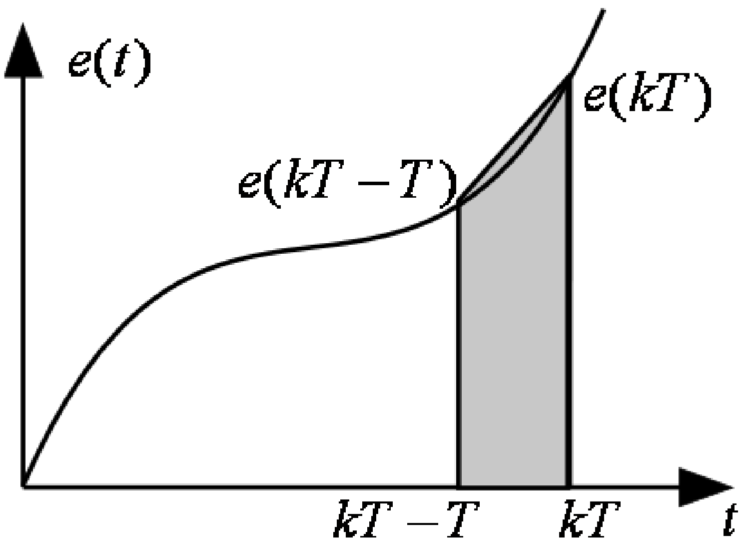
$$u(k) = u(k-1) + \frac{T}{2}[e(k-1) + e(k)]$$

$$U(z) = z^{-1}U(z) + \frac{T}{2}[z^{-1}E(z) + E(z)]$$

$$\frac{U(z)}{E(z)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$D(s) = \frac{a}{s+a}, D(z) = \frac{a}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

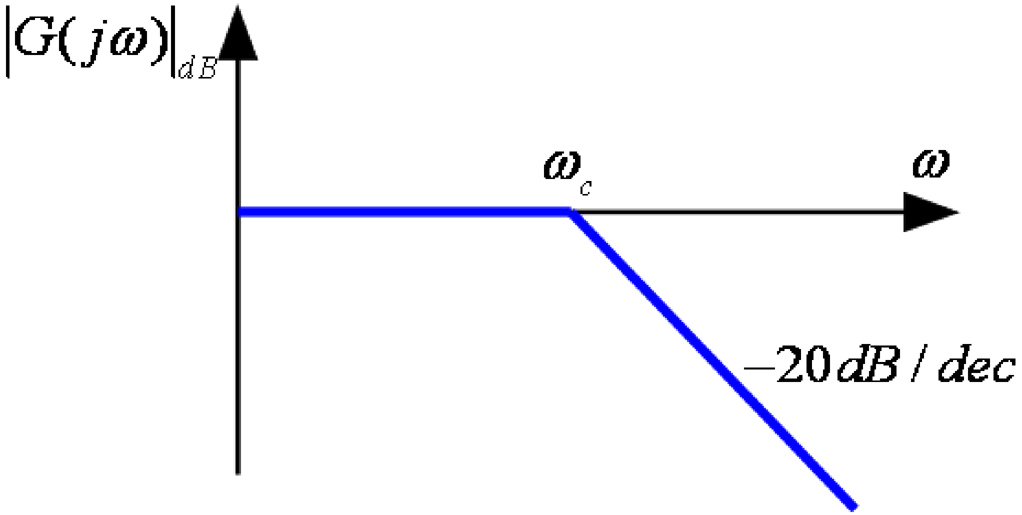
$$s \leftarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$



1st Order Digital Filter

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1 + s/\omega_c}$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 + \frac{1 - z^{-1}}{T\omega_c}} = \frac{T\omega_c}{T\omega_c + 1 - z^{-1}}$$



1st Order Digital Filter

$$(T\omega_c + 1 - z^{-1})Y(z) = T\omega_c U(z)$$

$$(T\omega_c + 1)Y(z) - z^{-1}Y(z) = T\omega_c U(z)$$

$$Y(z) = \frac{1}{(T\omega_c + 1)}\left[z^{-1}Y(z) + T\omega_c U(z)\right]$$

$$y(k) = \frac{1}{(T\omega_c + 1)}\left[y(k-1) + T\omega_c u(k)\right]$$

2nd Order Digital Filter

$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{Y(z)}{U(z)} = \frac{\omega_n^2}{\left(\frac{1-z^{-1}}{T}\right)^2 + 2\zeta\omega_n\left(\frac{1-z^{-1}}{T}\right) + \omega_n^2}$$